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Optimal risk transfer for agents with germs

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ABSTRACT

We introduce a new class of risk measures called generalized entropic risk measures (GERMS) that allow economic agents to have different attitudes towards different sources of risk. We formulate the problem of optimal risk transfer in terms of these risk measures and characterize the optimal transfer contract. The optimal contract involves what we call *intertemporal source-dependent quotient sharing*, where agents linearly share changes in the aggregate risk reserve that occur in response to shocks to the system over time, with scaling coefficients that depend on the attitudes of each agent towards the source of risk causing the shock. Generalized entropic risk measures are not dilations of a common base risk measure, so our results extend the class of risk measures for which explicit characterizations of the optimal transfer contract can be found.

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1. Introduction

Optimal risk transfer is concerned with finding "optimal" risk sharing agreements between economic agents. In one version of this problem, each agent has a utility function and the objective is to find risk sharing contracts that deliver a set of Pareto optimal utilities (Borch, 1960a,b, 1962; Buhlmann and Jewell, 1979; Burgert and Ruschendorf, 2006, 2008; Du Mouchel, 1968; Gerber, 1978). Another closely related version of this problem is to find risk sharing contracts which minimize the aggregate risk of the group as measured by the sum of risk measures over all agents (see for example Deprez and Gerber (1985) and more recently the papers of Barrieu and El Karoui (2004, 2005, 2009)). These problems have a long history in the insurance and economics literature (Arrow, 1953; Borch, 1960a,b, 1962; Buhlmann, 1980, 1984; Burgert and Ruschendorf, 2006, 2008; Deprez and Gerber, 1985; Du Mouchel, 1968; Gerber, 1978) and have recently attracted attention in the quantitative finance community (see for example Barrieu and El Karoui, 2004, 2005, 2009; Filipovic and Kupper, 2008; Heath and Ku, 2004; Jouini et al., 2008). In this regard, a well-known result attributed to Borch (1960a,b, 1962) is that the optimal transfer contract involves linear sharing of pooled losses if the risk measures of all agents are dilations of a common base measure (see also Barrieu and El Karoui, 2004, 2005, 2009; Deprez and Gerber, 1985; Du Mouchel, 1968; Gerber, 1978). In general, however, explicit construction of optimal contracts is not possible.

This paper makes several contributions to the literature on risk measures and optimal risk transfer problems. Firstly, we introduce a natural extension of the classical entropic risk measure (also known as an exponential premium in the insurance literature (Goovaerts et al., 2004; Goovaerts and Laeven, 2008)) which we call a *generalized entropic risk measure* (GERM) or *generalized exponential premium*.[1] A generalized entropic risk measure/generalized exponential premium allows for different risk attitudes towards different risk sources and reduces to classical entropic risk measure/exponential premium when attitudes towards all risk sources are the same. If a risk measure is interpreted as a cash reserve, then multiple risk attitudes come up if there are different rules governing the reserve that needs to be set aside for holdings in different markets (risk sources). In the context of insurance applications, it corresponds to different risk attitudes towards different sources of risk (e.g. home or auto insurance claims versus rare events such as earthquakes). Our second contribution is a characterization of the optimal risk transfer contract when agents adopt generalized entropic risk measures/exponential premiums in a market where losses are modeled by a pure jump

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[1] The terms *(generalized) entropic risk measure* and *(generalized) exponential premium* will be used interchangeably throughout this paper.

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process. The resulting contract involves what we call *intertemporal source-dependent quotient sharing*. Specifically, agents linearly share changes in the aggregate cash reserve that occur whenever there is a claim or shock to the system. These reserve injections are shared linearly between agents with scaling coefficients that depend on the attitudes of each agent towards the risk causing the shock (so home insurance claims are shared differently from auto claims which are shared differently from earthquake claims etc, depending on the risk attitudes of the insurance writers to each of these risk sources). The risk measures adopted by different agents are not necessarily dilations of the same common base so our results extend the class of risk measures for which an explicit characterization of the optimal contract can be found.

The outline of this paper is as follows. We survey classical results pertaining to optimal risk transfer and convex risk measures in Section 2. Generalized entropic risk measures/exponential premiums are introduced in Section 3 in the context of our pure jump stochastic model. The optimal risk transfer problem is studied in Section 4 where the notion of *intertemporal source-dependent quotient sharing* contracts is introduced. The optimal contract for the risk transfer problem is also shown to belong to this class. Several examples are presented in Section 5.

As a final note, though we have restricted ourselves to pure jump models, which are certainly of interest in risk management and insurance applications since they naturally model large losses and are sufficiently rich to model phenomena such as systematic risk as well as contagion effects, our results extend to other (e.g. Brownian) models as well as to other specifications of dynamic risk measures. These extensions will be reported elsewhere.

2. Classical results

Convex risk measures were first introduced (to our knowledge) in the paper of Deprez and Gerber (1985). Closely related (though not identical) notions were introduced by Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002) and analyzed using ideas from convex analysis. In this section we summarize some of these key results and introduce the problem of optimal risk transfer. The notion of inf-convolution (Rockafellar, 1970) and its relationship to optimal risk transfer will also be discussed (Barrieu and El Karoui, 2004, 2005, 2009). We recall the so-called entropic risk measure (or exponential premium) and its dual representation which forms the basis of our definition of a generalized entropic risk measure/generalized exponential premium which we introduce in the sequel.

2.1. Convex risk measures

We are primarily concerned with risk measures defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The set of all financial risky positions will be given by the set of all $\mathcal{F}$-measurable $\mathbb{P}$-a.s. bounded random variables $\mathcal{L}^\infty = \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$. All equalities and inequalities are taken to hold $\mathbb{P}$-a.s. The framework of this study will be that of Föllmer and Schied (2002) and Föllmer and Schied (2002), where the following definition can be found.

Definition 2.1 (*Convex Risk Measure*)**.** A map $\rho : \mathcal{L}^\infty \to \mathbb{R}$ is called a convex risk measure if, for all $\Psi, \Phi \in \mathcal{L}^\infty$, it satisfies the following properties:

(a) (Convexity) $\forall \lambda \in [0, 1]$, $\rho(\lambda\Psi + (1-\lambda)\Phi) \le \lambda\rho(\Psi) + (1-\lambda)\rho(\Phi)$;
(b) (Monotonicity) $\Psi \le \Phi \Rightarrow \rho(\Psi) \ge \rho(\Phi)$;
(c) (Translation invariance) $\forall a \in \mathbb{R}$, $\rho(\Psi + a) = \rho(\Psi) - a$.

A convex risk measure ρ is a coherent risk measure (Artzner et al., 1999) if it also satisfies:

(d) (Positive homogeneity) $\forall \lambda \ge 0$, $\rho(\lambda\Psi) = \lambda\rho(\Psi)$.

Since $\rho(\Psi + \rho(\Psi)) = 0$, the convex risk measure $\rho(\Psi)$ for a risky position Ψ can be interpreted as the reserve requirement for a risky position Ψ. It is well known that a convex risk measure which is continuous from above has the following dual representation (Föllmer and Schied, 2002).

Theorem 2.1. *Let ρ be a convex risk measure that is continuous from above* [2]*:*

$$\Psi_n \searrow \Psi \Rightarrow \rho(\Psi_n) \nearrow \rho(\Psi).$$

Then ρ has a dual representation

$$\rho(\Psi) = \sup_{\mathbb{Q}\in\mathcal{M}_1} \{\mathbb{E}_{\mathbb{Q}}[-\Psi] - \alpha(\mathbb{Q})\}, \quad \forall \Psi \in \mathcal{L}^\infty$$

where $\alpha : \mathcal{M}_1 \to \mathbb{R} \cup \{+\infty\}$ is a penalty function and $\mathcal{M}_1$ is the set of all probability measures on $(\Omega, \mathcal{F})$. If ρ is a coherent risk measure, then the dual representation is given by

$$\rho(\Psi) = \sup_{\mathbb{Q}\in\mathcal{M}_1} \{\mathbb{E}_{\mathbb{Q}}[-\Psi] | \alpha(\mathbb{Q}) = 0\}, \quad \forall \Psi \in \mathcal{L}^\infty.$$

2.2. Optimal risk transfer

The following is adapted from Barrieu and El Karoui (2009) and Deprez and Gerber (1985). Suppose there are two economic agents operating in the same risky universe modeled by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Agent i has risky position modeled by the random variable Ψ_i which is realized at the fixed time T and a convex risk measure ρ_i. Let Φ be an $\mathcal{F}$-measurable random variable corresponding to the derivative contract that is written by Agent 1 and sold to Agent 2 for the price π. If the contract (Φ, π) is sold by Agent 1, then his/her risky position will be $\Psi_1 - \Phi + \pi$, while Agent 2's resulting position will be $\Psi_2 + \Phi - \pi$. Agent 1's goal is to find the contract (Φ, π) which minimizes his/her resulting risk measure value, $\rho_1(\Psi_1 - \Phi + \pi)$. At the same time, the contract has to be such that it is acceptable to Agent 2 in the sense that $\rho_2(\Psi_2 + \Phi - \pi) \le \rho_2(\Psi)$; Agent 2 is no worse off by taking the other side of the deal. This leads to the optimization problem.[3]

$$\begin{aligned} &\inf_{\Phi,\pi} \rho_1(\Psi_1 - \Phi + \pi) \\ &\text{subject to:} \\ &\rho_2(\Psi_2 + \Phi - \pi) \le \rho_2(\Psi), \\ &\Phi \in \mathcal{L}^\infty, \ \pi \in \mathbb{R}. \end{aligned} \tag{1}$$

Translation invariance of risk measures implies that this problem can be written as

$$\begin{aligned} &\inf_{\Phi,\pi} \rho_1(\Psi_1 - \Phi) - \pi \\ &\text{subject to:} \\ &\rho_2(\Psi_2 + \Phi) + \pi \le \rho_2(\Psi) \\ &\Phi \in \mathcal{L}^\infty, \ \pi \in \mathbb{R} \end{aligned}$$

and it is clear that the inequality will be satisfied with strict equality at optimality. It follows that the optimal price satisfies

$$\pi^* = \rho_2(\Psi_2) - \rho_2(\Psi_2 + \Phi^*) \tag{2}$$

and the risk transfer problem is equivalent to

$$\inf_{\Phi\in\mathcal{L}^\infty} \rho_1(\Psi_1 - \Phi) - \{\rho_2(\Psi) - \rho_2(\Psi_2 + \Phi)\}.$$

Observing that $\rho_2(\Psi)$ is independent of the decision variable, this optimization problem is equivalent to finding (if possible) a contract Φ^* such that

[2] This condition is equivalent to a certain lower semi-continuity condition with respect to bounded pointwise convergence; see Föllmer and Schied (2002, Lemma 4.16) for a precise statement.

[3] Similar transfer problems to (1) can be formulated between economic agents with ρ_1 and ρ_2 being interpreted as utility functions. In particular, the reduction of the optimal transfer problem (1) to (3) can also be carried out if ρ_1 and ρ_2 are certainty equivalents of exponential utility.