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Optimal risk transfer under quantile-based risk measurers

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HIGHLIGHTS

- Find the optimal risk transfer among multiple insurance players under some quantile-based risk measure criteria.
- Develop a new constructive methodology to solve the well-known problem of optimal reinsurance.
- Find the set of all optimal solutions for the proposed optimisation problem.

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ABSTRACT

The classical problem of identifying the optimal risk transfer from one insurance company to multiple reinsurance companies is examined under some quantile-based risk measure criteria. We develop a new methodology via a two-stage optimisation procedure which not only allows us to recover some existing results in the literature, but also makes possible the analysis of high-dimensional problems in which the insurance company diversifies its risk with multiple reinsurance counter-parties, where the insurer risk position and the premium charged by the reinsurers are functions of the underlying risk quantile. Closed-form solutions are elaborated for some particular settings, although numerical methods for the second part of our procedure represent viable alternatives for the ease of implementing it in more complex scenarios. Furthermore, we discuss some approaches to obtain more robust results.

1. Introduction

The optimality problem of the risk transfer contract between two insurance companies within a one-period setting appears in different forms in the literature. The first attempts are attributed to Borch (1960) and Arrow (1963), where maximising the expected utility defines the optimality criterion. Further extensions have been developed for various decision criteria that depend on the risk measure choice (for example, see Van Heerwaarden et al., 1989; Young, 1999; Kaluszka, 2001, 2005; Verlaak and Beirlant, 2003; Kaluszka and Okolewski, 2008; Ludrovski and Young, 2009). Decisions based on two particular risk measures, *Value-at-risk (VaR)* and *Expected shortfall (ES)*, are considered by Cai et al. (2008), Cheung (2010), and Chi and Tan (2011). All of the aforementioned papers deal with the one-period model. The classical risk model setting has been successfully studied in the literature by Centeno and Guerra (2010) and Guerra and Centeno (2008, 2010), via maximisation of the adjustment coefficient.

There are two parties involved in a reinsurance contract: the *insurer* or *cedent*, who has an interest in transferring part of its risk, and the *reinsurer*. Let $X \geq 0$ be the total loss amount incurred during the duration of the insurance contract, with distribution function denoted by $F(\cdot)$ and survival function $\bar{F}(\cdot) = 1 - F(\cdot)$. In addition, the right end-point $x_F := \inf\{z \in \Re : F(z) = 1\}$ of the loss distribution can be either finite or infinite. The reinsurance company agrees to pay $R[X]$, the amount by which the entire loss exceeds the insurer amount, $I[X]$. Thus, $I[X] + R[X] = X$. There are many possible reinsurance arrangements, which depend on the particular choice of the insurer and reinsurer of sharing the premiums and underwritten risks. For example, the liabilities are shared in a fixed proportion under *proportional* reinsurance, and therefore $I[X] = cX$, where $0 < c < 1$ is a constant. Another common arrangement is *Stop-loss* reinsurance contracts, for which the cedent loss is limited to a fixed amount, M, known as the *retention limit*. The net amount paid by the insurer is therefore given by $\min\{X, M\} := X \wedge M$.

The reinsurer premium, $\mathbf{P}\big(R[X]\big)$, is usually assumed to satisfy $\mathbf{P}\big(R[X]\big) \geq \mathbf{E}\big(R[X]\big)$, since otherwise the risk bearer would become insolvent almost surely. Obviously, the total insurer loss becomes

$$\mathbf{L}\big(R[X]\big) := I[X] + \mathbf{P}\big(R[X]\big).$$

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The aim of this paper is to identify the optimal arrangement that lays the cedent in the best possible situation towards the risk. That is, we intend to minimise $\varphi_I(\mathbf{L}(R[X]))$ over a set of feasible reinsurance contracts, where φ_I represents a measure of the risk taken by the insurer. Motivated by the standard regulatory requirements developed within the insurance industry, the risk exposures are measured via VaR and ES risk measures. The VaR of a generic loss variable Z at a confidence level α, $VaR_\alpha(Z)$, represents the minimum amount of capital that makes the insurance company to be solvent at least α% of the time. The mathematical formulation is then given by

$$VaR_\alpha(Z) := \inf\{z \geq z_0 : \Pr(Z \leq z) \geq \alpha\},$$

where $z_0 := \sup\{z \in \Re : \Pr(Z \leq z) = 0\}$ represents the left end point of the distribution of Z. The ES at a confidence level α, $ES_\alpha(Z)$, evaluates the expected loss amount incurred among the worst $(1-\alpha)$% scenarios. Clearly, the ES represents a more conservative risk measure than the VaR, especially in the situation in which both risk quantifications are made at the same confidence level. The ES has multiple formulations in the literature, and the comprehensive papers on this topic of Acerbi and Tasche (2002) and Hürlimann (2003) may help in clarifying the differences between equivalent representations. We only refer to the next definition:

$$\begin{aligned} ES_\alpha(Z) &:= \frac{1}{1-\alpha}\int_\alpha^1 VaR_s(Z)ds \\ &= VaR_\alpha(Z) + \frac{1}{1-\alpha}\mathbf{E}\big(Z - VaR_\alpha(Z)\big)_+, \end{aligned} \tag{1.1}$$

where $(z)_+ = \max\{z, 0\}$. Interestingly, this risk measure is a special case of the Haezendonck–Goovaerts class, which was introduced many years ago by Haezendonck and Goovaerts (1982). Further details can be found in Bellini and Rosazza Gianin (2012), Goovaerts et al. (2004, 2012), and the references therein.

As a result of the translation invariance property of the three risk measures, the following holds:

$$\varphi_I\Big(\mathbf{L}\big(R[X]\big)\Big) = \varphi_I\big(I[X]\big) + \mathbf{P}\big(R[X]\big).$$

In order to avoid potential moral hazard issues related to the reinsurance arrangement, the set of feasible contracts is given by

$$\mathcal{F} := \Big\{R(\cdot) : I(x) = x - R(x) \text{ and } R(x) \text{ are non-decreasing functions}\Big\}.$$

It is useful to note that $R \in \mathcal{F}$ implies that the functions I and R are Lipschitz functions with unit constants, i.e. $|I(y) - I(x)| \leq |y - x|$ and $|R(y) - R(x)| \leq |y - x|$ are true for all $x, y \geq 0$. Therefore, our optimisation problem is reduced to

$$\min_{R\in\mathcal{F}}\Big\{\varphi_I\big(X\big) - \varphi_I\big(R[X]\big) + \mathbf{P}\big(R[X]\big)\Big\}, \tag{1.2}$$

since $I[X]$ and $R[X]$ are co-monotone random variables (for details, see Dhaene et al., 2002a,b; Denuit et al., 2005). In other words, the co-monotonicity property implies that

$$\varphi_I\big(X\big) = \varphi_I\big(I[X]\big) + \varphi_I\big(R[X]\big).$$

It should be noted that we give explicit derivations whenever the insurer's risk position is evaluated via the ES and VaR risk measures, but our procedure can be easily applied to any risk measure that is a function of the risk quantile. In Section 4, we will illustrate this for a more robust risk measure, namely the *Truncated tail value-at-risk (TrTVaR)*. A large class of such risk measures is given by

$$M_\Phi(Z) := \int_0^1 VaR_s(Z)\Phi(s)\, ds,$$

where the function $\Phi(\cdot)$ has certain properties. This class is known as the *distorted* (see Wang and Young, 1998; Jones and Zitikis, 2003) and *spectral* (see Acerbi, 2002) class of risk measures, respectively. Therefore, our procedure is widely applicable to situations in which the insurer risk position is evaluated via many well-known risk measures.

A common premium principle used in practice is the *expected value* principle. That is, the reinsurer premium is loaded as follows:

$$\mathbf{P}\big(R[X]\big) = (1+\rho)\mathbf{E}\big(R[X]\big),$$

where $\rho > 0$ is known as the *security loading factor*. The main problem defined by Eq. (1.2) has been previously investigated in the literature when the insurer has set the VaR or ES as the baseline risk measure. Cai et al. (2008) and Cheung (2010) found the optimal reinsurance contract over a class of convex functions $R(\cdot)$. Note that both papers assume $F(\cdot)$ to be a strictly increasing and continuous function, and therefore their *Conditional tail expectation* evaluation coincides with that of the ES. Chi and Tan (2011) elaborate a two-stage optimisation procedure to solve the VaR and ES problem, where the solution of the first problem is given a priori.

Once again, our procedure can be applied to other premium principles that are functions of the underlying risk quantile. Section 3 contains some distorted premium principles that should help in supporting our statement. The expected value principle is chosen only for the sake of exposition, and also to recover some existing results from the literature.

So far, it has been implicitly assumed that the insurer may transfer the risk to only one reinsurance counter-party. More realistic situations involve multiple reinsurance risk transfers available on the market. It is likely that each reinsurer has its own pricing model, and the cedent may choose to transfer specific layers from the total risk to competitive companies. Therefore, the insurer may improve the diversification gain by sharing the loss with multiple reinsurance counter-parties. At this point, it is worth mentioning that all previously mentioned papers assume the reinsurance market to consist of only one agent.

In this paper, we present a general two-stage based algorithm for identifying optimal arrangements when the insurer risk is diversified through multiple reinsurance agreements. More specifically, we investigate some optimisation problems within a trivariate risk transfer setting with different assumptions regarding the premium principles used by the two reinsurance counter-parties. To our knowledge, the proposed method represents the only choice for an insurance company of taking advantage of different available reinsurance pricing schemes on the market. However, this situation tends to be cumbersome if more than two reinsurance companies are potential players in the risk transfer game. We derive closed-form solutions for two particular scenarios, but numerical methods are required for solving the second-stage problem in order to overcome this potential computational issue. In the one-dimensional reinsurance market case, our approach can also be viewed as an alternative solution to existing methods, but we clearly derive the solution of the first optimisation problem. However, our proposed method has the advantage of solving more complex situations, whenever the insurer's risk position and the reinsurer premiums are functions of the risk quantile. Note that our approach can be extended to many more reinsurance companies than just two.

The rest of the paper is organised as follows. The next section gives a different perspective for the ideal ES-based and VaR-based reinsurance arrangement under the assumption that there is only one reinsurance company on the market, for which some of the results can already be found in the existing literature. The third section illustrates some VaR-based and ES-based optimal decisions for an insurer, whenever the cedent shares the risk with two other reinsurance agents. The fourth section deals with robust estimation of the optimal reinsurance arrangement. In this section we will also obtain the ideal TrTVaR-based reinsurance