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Mathematical and Computer Modelling 38 (2003) 1399–1408

MATHEMATICAL AND COMPUTER MODELLING

www.elsevier.com/locate/mcm

Study on Option Pricing in an Incomplete Market with Stochastic Volatility Based on Risk Premium Analysis

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Abstract—We focus on an option pricing mechanism in a market, for which the underlying asset has a stochastic volatility. This model generally belongs to the class of incomplete market models, and hence, a no-arbitrage option price is not uniquely determined. However, in the actual market, options are traded at some price. This fact may suggest that the market participant compromises on the risk caused by stochastic volatility and that the balance of risk aversion of sellers and buyers determines the market price. To analyze this mechanism, we introduce a concept of the risk premium for stochastic volatility (RPSV) and develop a method to estimate the RPSV implied in actual option prices. In the method, homogeneous RPSV is not required to allow the segmentation of the market. Estimated RPSV of Nikkei 225 options are almost positive, and certainly depend on the strike price and time to maturity.

Keywords—Option pricing, Stochastic volatility, Incomplete market model, Risk premium, Martingale, Estimation, Nikkei 225 options.

1. INTRODUCTION

When trading an option, it is necessary to calculate its appropriate value whatever the purpose of trading might be, for example hedging, speculation, or arbitrage. Many researchers have focused on option pricing theories and their application in practice. Nowadays, most of the practical pricing models are based on the Black and Scholes' option pricing model, hereafter BS model. However, the BS model requires many assumptions, e.g., geometric Brownian motion of the underlying asset price process, constant risk free interest rate, continuous and frictionless trading, and so on; and any of these assumptions can fail to hold in the actual market. On the other hand, a lot of empirical research suggests that the gap between these assumptions and the actual market is not negligible. First of all, the assumption of constant volatility should be

Comments from Dr. M. Ikeda and researchers in MTB Investment Technology Institute Co., Ltd. are gratefully acknowledged.
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0895-7177/03/$ - see front matter Typeset by AMS-TEX
doi: 10.1016/S0895-7177(03)00354-6

removed, because volatility plays a very important role in option pricing and itself could fluctuate over even a comparatively short term, say one to three months.

Many general pricing models not requiring the assumption of constant volatility have been developed. These can be classified into the following groups.

(a) Volatility is described as a function of time and the price of the underlying asset and the function is calibrated to fit the shape of implied volatility [1–4].
(b) Volatility's dynamics is modeled by a time series model such as GARCH, E-GARCH, and so on, [5,6].
(c) Volatility's fluctuation is described as a stochastic differential equation in continuous time [7–10].

In this paper, we take the third approach and assume that there is an independent randomness in the fluctuation of the underlying asset's volatility, and focus on the risk premium for stochastic volatility implied in option prices.

2. OPTION PRICING UNDER STOCHASTIC VOLATILITY

2.1. Stochastic Volatility Model

We consider a market which contains a risk-free asset and a risky stock. For the dynamics of the stock, we assume a typical stochastic volatility model, hereafter SV model,

$$dS_t = S_t \left[b(S,\sigma,t)dt + \sqrt{\sigma_t} dW_{1,t} \right], \tag{1}$$

$$d\sigma_t = \sigma_t \left[\mu(S,\sigma,t)dt + \gamma(S,\sigma,t)\rho_t \, dW_{1,t} + \gamma(S,\sigma,t)\sqrt{1-\rho_t^2}\, dW_{2,t} \right]. \tag{2}$$

The model is defined on the filtered probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}\}_{0\le t\le T}$, $\mathcal{F} = \mathcal{F}_T$ generated by the two-dimensional standard Brownian motion $\mathbf{W}_t = \{(W_{1,t}, W_{2,t})^*\}_{0\le t\le T}$. Both the stock price process $\{S_t\}_{0\le t\le T}$ and its volatility process $\{\sigma_t\}_{0\le t\le T}$ are $\mathbf{R}^+$-valued $\mathcal{F}_t$-adapted processes. Here, ρ_t is the correlation process, such that $|\rho_t| \le 1$ for all $t \in [0,T]$. $b(S,\sigma,t)$, $\mu(S,\sigma,t)$, and $\gamma(S,\sigma,t)$ are uniformly bounded $\mathcal{F}_t$-adapted processes on $[0,T] \times \Omega$.

We also set the following assumptions about the market (as in the BS model):

- no arbitrage opportunity among each option, stock, and risk-free asset,
- frictionless market,
- continuous trading,
- no limitations on selling or buying the stock,
- deterministic risk free interest rate.

2.2. Martingale Valuation of Option Prices

We consider European options written on the stock as $\mathcal{F}_T$-measurable random variables described by $\mathbf{R}^+$-valued functions of the stock price at the maturity date T, i.e., $B = g(S_T)$. Under the no-arbitrage assumption among an option, stock, and risk-free asset, the option price φ_t is evaluated by means of martingale valuation [11] as conditional expectation under a probability measure Q;

$$\varphi_t = E_Q\left[\frac{\beta_T B}{\beta_t} \,\middle|\, \mathcal{F}_t \right], \tag{3}$$

where β_t is the discount process $\beta_t := \exp(-\int_0^t r_s \, ds)$, $t \in [0,T]$.