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Research announcement

# Option pricing in incomplete markets

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ABSTRACT

Expected utility maximization is a very useful approach for pricing options in an incomplete market. The results from this approach contain many important features observed by practitioners. However, under this approach, the option prices are determined by a set of coupled nonlinear partial differential equations in high dimensions. Thus, it represents numerous significant difficulties in both theoretical analysis and numerical computations. In this paper, we present accurate approximate solutions for this set of equations.


## 1. Introduction

The well-known Black–Scholes formula [1] for option pricing is only valid under the assumption of complete markets. Empirical study shows that volatility is stochastic and this leads to an incomplete market. Stochastic volatility is a common situation which leads to an incomplete market. The Heston model [2] of stochastic volatility has been widely adopted in modern finance, especially in pricing options. Hodges and Neuberger [3] first introduced a notion of *utility indifference price*. At this particular price, an agent aiming to maximize his/her expected utility is indifferent as to whether or not to purchase an additional unit of the option. Davis [4] suggested the notion of *fair price*, which is the utility indifference price for holding an infinitesimal position in options. Yang [5] and Stoikov [6] have further extended the concepts of fair price and utility indifference price to portfolios. They carried out a detailed study of option pricing in a stochastic volatility setting under an exponential utility function. However, this represents an extremely challenging numerical problem, since it involves coupled nonlinear partial differential equations (PDEs) in high dimensions. Therefore, it is desirable to obtain approximate analytical solutions for these equations. This is the main contribution of this paper.

## 2. Main results

We consider a market that consists of a riskless asset, a risky stock of price $s$ and $N$ European options $f^i$ on $s$, ($i = 1, 2, \ldots, N$), with the payoff $f^i(T_i, s, v) = (s - k_i)^+$ for a call and $f^i(T_i, s, v) = (k_i - s)^+$ for a put. Here $k_i$ is the strike price and $T_i$ is the maturity date. The stock dynamics $s$ is modeled by

$$ds = \nu(v)s\,dt + \sqrt{v_t}s\,dB_t^s, \tag{1}$$

where $\nu$ is the drift, and $v_t$, the instantaneous variance, is governed by Heston's volatility process

$$dv_t = \kappa(\theta - v)\,dt + \xi v^{\frac{1}{2}}\,dB_t^v. \tag{2}$$

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The two standard Brownian motions $dB_t^s$ and $dB_t^v$ are correlated ($dB_t^s\,dB_t^v = \rho\,dt$), and $\kappa, \theta, \xi$ and $\rho$ are assumed constant. Since one can introduce discounted financial instruments to eliminate the effect of the constant interest rate $r$, we set $r = 0$ in this paper. We assume that the stock may pay a constant continuous dividend yield at the rate $q$, and that the trading allows unlimited lending and borrowing. We consider a portfolio of wealth $w_t$, which holds $n_0$ shares of stock and $n_i$ units of option $f^i$ at time $t$. The investor manages his/her portfolio based on the maximization of the expected exponential utility function $U(w) = -e^{-\gamma w}/\gamma$, where $\gamma$ is a positive parameter modeling an investors attitude toward risk.

This approach leads to the following set of nonlinear partial differential equations, which determine the fair prices $f^i$ of European options and the portfolio-indifference price $h$ [5]:

$$f_t^i - qsf_s^i + [\kappa(\theta - v) - \rho\chi\xi v - \gamma(1-\rho^2)\xi^2 v(\bar{\phi}_v + h_v)]f_v^i + \mathcal{O}_2 f^i = 0, \tag{3}$$

$$\begin{aligned} &h_t - qsh_s + [\kappa(\theta - v) - \rho\chi\xi v - (1-\rho^2)\xi^2 v\gamma\bar{\phi}_v]h_v + \mathcal{O}_2 h \\ &\quad - \frac{1}{2}\gamma(1-\rho^2)\xi^2 v h_v^2 + \sum_{i=1}^{N} n_i f^i(T_i, s, v)\delta(t - T_i) = 0 \end{aligned} \tag{4}$$

with the final conditions $f^i(T_i, s, v) =$ payoff and $h(T, s, v) = 0$, where $\nu(v) + q = \chi v$, $T = \max\{T_i\}$ and $\mathcal{O}_2\psi(t, s, v) \doteq \frac{1}{2}vs^2\psi_{ss} + \frac{1}{2}\xi^2 v\psi_{vv} + \rho\xi vs\psi_{sv}$. Here $\delta(*)$ is the Dirac delta functions to ensure continuity when time passes through any maturity date $T_i$, $\bar{\phi}$ is determined by $\gamma\bar{\phi}(\tau, v) = k(\tau) + l(\tau)v$, and

$$l(\tau) = \frac{2[e^{\sqrt{\Delta}\tau} - 1]}{(\sqrt{\Delta} + \beta)e^{\sqrt{\Delta}\tau} + (\sqrt{\Delta} - \beta)}\Theta, \qquad k(\tau) = \frac{4\kappa\theta\Theta}{\Delta - \beta^2}\left[\ln\frac{(\sqrt{\Delta} + \beta)e^{\sqrt{\Delta}\tau} + (\sqrt{\Delta} - \beta)}{2\sqrt{\Delta}} - \frac{\beta + \sqrt{\Delta}}{2}\tau\right],$$

where $\tau = T - t$, $\beta = \kappa + \rho\xi\chi$, $\Theta = \frac{1}{2}\chi^2$ and $\Delta = \beta^2 + 2(1-\rho^2)\xi^2\Theta > 0$.

Eqs. (3) and (4) are coupled high-dimension nonlinear PDEs. There are no closed-form solutions. Numerical computations have been the main tools for solving these equations. However, numerical computations are difficult due to nonlinearity and time consuming due to the high dimensionality. We now present accurate approximate solutions for these equations. We consider that the volatility of volatility $\xi$ is small. So we replace $\xi$ by $\epsilon\xi$ ($0 < \epsilon \ll 1$). Thus, Eqs. (3) and (4) can be rewritten as

$$(\mathcal{D}_0 + \epsilon\mathcal{D}_1 + \epsilon^2\mathcal{D}_2)f^i - \epsilon^2 C(v)h_v f_v^i = 0, \tag{5}$$

$$(\mathcal{D}_0 + \epsilon\mathcal{D}_1 + \epsilon^2\mathcal{D}_2)h - \frac{1}{2}\epsilon^2 C(v)h_v^2 = 0, \tag{6}$$

where $C(v) = \gamma(1-\rho^2)\xi^2 v$ and the operators $\mathcal{D}_0$, $\mathcal{D}_1$ and $\mathcal{D}_2$ are defined as

$$\mathcal{D}_0 = \frac{\partial}{\partial t} - qs\frac{\partial}{\partial s} + \kappa(\theta - v)\frac{\partial}{\partial v} + \frac{1}{2}vs^2\frac{\partial^2}{\partial s^2},$$

$$\mathcal{D}_1 = -\rho\xi\chi v\frac{\partial}{\partial v} + \rho\xi vs\frac{\partial^2}{\partial s\partial v}, \qquad \mathcal{D}_2 = \frac{1}{2}\xi^2 v\frac{\partial^2}{\partial v^2} - (1-\rho^2)\xi^2 vl(v)\frac{\partial}{\partial v}.$$

We construct expansions for $f^i$ and $h$ in terms of the power of $\epsilon$:

$$f^i(t, s, v) = f^{i(0)}(t, s, v) + \epsilon f^{i(1)}(t, s, v) + \epsilon^2 f^{i(2)}(t, s, v) + \cdots, \tag{7}$$

$$h(t, s, v) = h^{(0)}(t, s, v) + \epsilon h^{(1)}(t, s, v) + \epsilon^2 h^{(2)}(t, s, v) + \cdots. \tag{8}$$

Inserting these expressions into Eqs. (6) and (7), we obtain

$$\mathcal{D}_0 f^{i(m)} = -\mathcal{D}_1 f^{i(m-1)} - \mathcal{D}_2 f^{i(m-2)} - C(v)h_v^{(m-2)}f_v^{i(m-2)} \tag{9}$$

$$\mathcal{D}_0 h^{(m)} + \delta_{0m}\sum_{i=1}^{N} n_i f^{i(m)}(T_i, s, v)\delta(t - T_i) = -\mathcal{D}_1 h^{(m-1)} - \mathcal{D}_2 h^{(m-2)} + \frac{1}{2}C(v)(h_v^{(m-2)})^2 \tag{10}$$

with the final conditions $f^{i(m)} = \delta_{0m}\cdot$ payoff and $h^{(m)} = 0$, where $f^{i(m)} = 0$ and $h^{(m)} = 0$ when $m < 0$. Here $\delta_{0m}$ is the Kronecker delta function with $\delta_{0m} = 1$ for $m = 0$, and $\delta_{0m} = 0$ for $m \neq 0$. For the zeroth-order terms ($m = 0$), the right hand sides of Eqs. (9) and (10) are zero. After applying the Feynman–Kac formula, the zeroth-order solutions, $f^{i(0)}$ and $h^{(0)}$, are expressed by

$$\begin{cases} \text{Call: } f^{i(0)}(t, s, v) = se^{-q\tau_i}N(d_i^+) - k_i N(d_i^-) \\ \text{Put: } f^{i(0)}(t, s, v) = k_i N(-d_i^-) - se^{-q\tau_i}N(-d_i^+) \end{cases}, \qquad h^{i(0)}(t, s, v) = \sum_{i=1}^{N} n_i f^{i(0)}(t, s, v), \tag{11}$$

where $\tau_i = T_i - t$, $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2}dz$, $d_i^{\pm} = \left[\log(se^{-q\tau_i}/k_i) \pm \frac{1}{2}\sigma_i^2(t)\right]/\sigma_i(t)$ and $\sigma_i^2(t) = \theta(T_i - t) - (\theta - v)(1 - e^{-\kappa(T_i - t)})/\kappa$. Knowing the zeroth-order solutions, the right hand sides of Eqs. (9) and (10) allow one to obtain the first-order solutions ($m = 1$) by applying the Feynman–Kac formula. This procedure can be repeated at higher orders and gives all higher-order solutions.