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Factorization of European and American option prices under complete and incomplete markets ☆

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Abstract

In a standard option-pricing model, with continuous-trading and diffusion processes, this paper shows that the price of one European-style option can be factorized into two intuitive components: One robust, X_0, which is priced by arbitrage, and a second, Π_0, which depends on a risk orthogonal to the traded securities. This result implies the following: (1) In an incomplete market, these parts represent the price of a hedging portfolio, which is *unique*, and a premium, which depends *only* on the risk premiums associated with the residual risk, respectively. (2) In a complete market, it allows factoring the contribution of the different sources of risk to the final option price. For example, in a stochastic volatility model, we can quantify the impact on the option price of volatility risk relative to market risk, Π_0 and X_0, respectively. Hence, certain misspricings in option markets can be directly related to the premium, Π_0. (3) Moreover, these results extend to American securities, which have a third component – an additional early-exercise premium.

JEL classification: G12; G13

Keywords: Option-pricing; Incomplete markets; Dynamic hedging; Option price factorization; Stochastic volatility; American options

1. Introduction

In an incomplete market there is not a replicating portfolio for those securities that are not spanned, and thus, one cannot apply the ‘‘law of one price’’ and obtain a unique solution. On the contrary, there are upper and lower arbitrage bounds, which contain the non-arbitrage prices (Merton, 1973). One must make further assumptions to select one of these prices or to constrain the arbitrage bounds.[1]

This paper shows that the price of a European-style security can be decomposed into the price of a hedging portfolio plus a premium. More precisely, for any non-spanned security, we do not provide a specific price, but show that any arbitrage-free price C_0 can be factored into

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[1] Including the use of utility maximization arguments (Merton, 1976; Rubinstein, 1976); to consider prices of risk associated with non-traded state variables (Heston, 1993); to compute an optimal hedging portfolio, whose price is the desired incomplete market price (Merton, 1998); to use a risk/reward criterion, such as the gains-to-losses ratio (Bernardo and Ledoit, 2000) or the Sharpe ratio (Cochrane and Saá-Requejo, 2000). See also Carr et al. (2001) and Cerný (2003). See Broadie and Detemple (2004, p. 1151) for a recent survey and more references to the literature.

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two components: $C_0 = X_0 + \Pi_0$. The first component, X_0, is the price of a hedging portfolio and is unique, as in a complete market, and, therefore, does not depend on C_0. The second one, Π_0, is the premium, which depends only on the stream of risk premiums associated with the residual risk, and is implicit in C_0.

The price factorization implies that we can see an option price as the sum of two separate parts, focusing on the premium Π_0. These results are simple and intuitive, but have not been explicitly written out as we do here. Moreover, these results can be applied to option prices in a complete market and extend to American-style securities as well. The decomposition holds in the standard asset-pricing model, which assumes continuous-trading and diffusion processes. These results follow from the fact that the option payoff can be divided into two orthogonal components: one spanned by the traded securities and a second orthogonal to them. Let us illustrate these results in a simple one-period model, which also clarifies the assumptions underlying them.

1.1. A one-period example

We price a security C_1. There are two traded securities, a riskless bond and a risky asset with prices 1 and $x_0 = 1$ and payoffs $R > 0$ and x_1, respectively. We assume that $C_1 \sim \mathcal{N}(\mu_c, \sigma_c^2)$ and $x_1 \sim \mathcal{N}(\mu_x, \sigma_x^2)$ are Gaussian (μ is the mean and σ^2 is the variance), so that minimizing the variance is the optimal and unique hedging criterion. Let ρ, with $|\rho| < 1$, be the correlation between C_1 and x_1. Then, the beta portfolio $h_1^* = \sigma_x^{-1}\sigma_c\rho$ minimizes the variance of $Y_1 = h_0R + h_1x_1 - C_1$, where $h_0^* = \frac{1}{R}(\mu_c - h_1^*\mu_x)$. The residual risk verifies that $Y_1 \sim \mathcal{N}(0, (1-\rho^2)\sigma_c^2)$ and $E[Y_1x_1] = 0$. We denote by $X_0 = h_0^* + h_1^*$ the price of this hedging portfolio.

Let C_0 be the price of the security C_1. First, $C_1 - Y_1$ is the spanned payoff of C_1, since the hedging criterion is unique, and X_0 is its corresponding price. Therefore, $C_0 - X_0$ is the risk premium associated with the unspanned payoff or residual risk Y_1, and holds for every price C_0. Second, it is a separate matter to determine the magnitude of this premium or how it is invested (i.e., $C_0 - X_0$ and α, respectively), where the final residual risk is given by $Y_1 + (C_0 - X_0)(R + \alpha(x_1 - R))$.

The arguments are similar in a dynamic model. Consider, for example, Heston's (1993) model, which depends on a non-traded variable, stochastic volatility. Assume that the only traded securities are the market portfolio and a bank account. Hence, the market is incomplete. By using dynamic spanning, it is possible to decompose the option payoff into two components: one that depends on market risk only, and that, therefore, can be hedged and priced by arbitrage. And a second component, which is orthogonal to the market and depends on stochastic volatility. We show that the price of this second component depends only on the stream of volatility risk premiums.

The option price factorization, $C_0 = X_0 + \Pi_0$, has interesting applications for option-pricing, which we describe next. First, it is easy to define an upper and a lower bound in an incomplete market, because the price of the hedging portfolio is the same for both bounds. For example, if one considers positive (negative) risk premiums for the upper (lower) bound, the term $\Pi_0 > 0$ ($\Pi_0 < 0$). Therefore, it is also easy to constrain the arbitrage bounds by constraining the risk premiums.

Second, the factorization is derived by using the risk-neutral measure that assigns zero risk premiums to the orthogonal risk, and applying Feynman–Kac theorem. Under this measure, the discounted upper (lower) bound is a super-martingale (sub-martingale) if the risk premiums are positive (negative). The discounted price of the hedging portfolio is the martingale component. These results are related to Ross (1978) and Harrison and Kreps (1979), but in incomplete markets.

As another application, consider the problem of pricing a portfolio of N securities. The price of this portfolio and the sum of the N individual prices can only differ in an incomplete market. The factorization implies that this difference is due to the valuation of the residual risk, since the price of the hedging portfolio is the same in both cases. If there is some diversification in the portfolio of the N securities (e.g., from having offsetting positions), this portfolio can be cheaper.

The decomposition can be applied to a complete market: to factor the contribution of the different sources of risk to the final option price. For example, in a stochastic volatility model, we can quantify the price impact of stochastic volatility relative to market risk, Π_0 and X_0, respectively. Note that Π_0 can be explicitly computed as the difference of two option prices; i.e., $C_0 - X_0$. Accordingly, the percent option premium Π_0/X_0 is a simple measure of mispricing (see Ibáñez, 2006).

The decomposition also applies to American-style securities. Extending previous results of complete markets (e.g., Kim, 1990; Carr et al., 1992, and Broadie and Detemple, 2004), an American option is divided into three components: a risk premium, the price of the hedging portfolio of the equivalent European option, and an extra early-exercise premium. We indeed show two different factorizations. These are novel results as, unlike European options, American options in incomplete markets have received little attention.

As noted above, the premium, Π_0, depends on the risk premiums associated with the residual risk. However, differing from a complete market, these risk premiums do not need to depend on "prices of risk" to avoid arbitrage. This is a flexibility which could be used to better fit volatility smiles.[2] If one associates a price of risk with every non-traded variable, the approach reduces to risk-neutral pricing. One can show that Merton (1998) and Cochrane and Saá-Requejo (2000) present two approaches where

[2] E.g., Duffee (2002) and Duarte (2004) specify flexible prices of risk for (complete markets) term structure models.