Journal of Financial Economics 63 (2002) 161–210
JOURNAL OF Financial ECONOMICS
www.elsevier.com/locate/econbase

Simulated likelihood estimation of diffusions with an application to exchange rate dynamics in incomplete markets[☆]

Michael W. Brandt[a,*], Pedro Santa-Clara[b]

[a] *The Wharton School, University of Pennsylvania, Philadelphia, PA 19104, USA*
[b] *The Anderson School at UCLA, Los Angeles, CA 90095, USA*

Received 28 July 1999; received in revised form 1 May 2001

Abstract

We present an econometric method for estimating the parameters of a diffusion model from discretely sampled data. The estimator is transparent, adaptive, and inherits the asymptotic properties of the generally unattainable maximum likelihood estimator. We use this method to estimate a new continuous-time model of the joint dynamics of interest rates in two countries and the exchange rate between the two currencies. The model allows financial markets to be incomplete and specifies the degree of incompleteness as a stochastic process. Our empirical results offer several new insights into the dynamics of exchange rates. © 2002 Elsevier Science B.V. All rights reserved.

JEL classificaion: G12; G15; C32

Keywords: Estimation of diffusions; Exchange rates; Incomplete markets

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[☆] This paper and earlier versions circulated under different titles benefited from the comments of Yacine Aït-Sahalia, João Amaro de Matos, Domenico Cuoco, Lars Peter Hansen, Lane Hughston, Frank de Jong, Bob Kimmel, Lars Nielsen, Monika Piazzesi, Matt Richardson, Jesús Saá-Requejo, José Scheinkman, the editor Bill Schwert, Avanidhar Subrahmanyam, Ross Valkanov, and Ivo Welch. We are grateful to Chris Jones, whose detailed and knowledgeable comments went far beyond the usual call of duty. We also thank the seminar participants at the 2001 conference on randomized algorithms in finance of the Mathematical Sciences Research Institute, 2000 meeting of the American Finance Association, Carnegie-Mellon University, INSEAD, London Business School, NBER asset pricing group, NBER forecasting and empirical methods group, Northwestern University, Ohio State University, UCLA, University of Michigan, University of Pennsylvania, University of Rochester, and University of Texas for their comments and suggestions. Bing Han, Canlin Li, Jiang Luo, and Dmitry Livdan provided excellent research assistance. We are responsible for any remaining errors.

*Corresponding author. Tel.: +1-215-898-3609; fax: +1-215-898-6200.
E-mail address: brandtm@wharton.upenn.edu (M.W. Brandt).

PII: S0304-405X(01)00093-9

1. Introduction

Many theoretical models in economics and finance are formulated in continuous time as a diffusion or a system of diffusions, although the data these models describe can only be sampled at discrete points in time. The popularity of diffusions creates a need for effective econometric methods for estimating continuous-time models. In this paper, we present a simulation-based estimator of the parameters of a diffusion or a system of diffusions from discretely sampled data. The estimator is transparent, adaptive, and inherits the asymptotic properties of the generally unattainable maximum likelihood estimator.

Theorists in various areas prefer the continuous-time diffusion setting because of the tractability offered by Itô calculus. In financial models the continuous-time setting also plays a conceptual role. Since Black and Scholes (1973) and Merton (1971), many asset pricing models have assumed dynamic trading in continuous time to allow markets to be complete and hence derivative payoffs or consumption trajectories to be spanned, even when there exists a continuum of states and only a few traded securities. Diffusions are attractive from a statistical perspective because they are fully characterized by their instantaneous mean and variance. Also, the continuous time setting breaks the link between the model and the sampling frequency of the data, which is particularly important for nonlinear models that have different distributional characteristics at different sampling frequencies. For example, consider a GARCH model, which is a nonlinear process with Gaussian transitions, specified at a daily frequency. With daily data, maximum likelihood estimation of the model is straightforward. With weekly data, in contrast, the transitions between observations are no longer Gaussian and maximum likelihood estimation is much more complicated.

As for any parametric model, maximum likelihood is the preferred method for estimating the parameters of a diffusion. Unfortunately, exact maximum likelihood estimation is only possible in a few special cases when the distribution of the discretely sampled data is known. In particular, the distribution is know explicitly for diffusions with linear mean and constant or proportional variance (Chen and Scott, 1993; Pearson and Sun, 1994), and it is known up to an inversion of the characteristic function for all affine jump-diffusions (Singleton, 2001). In most cases, however, exact maximum likelihood estimation is impossible because the likelihood function of the model cannot be evaluated explicitly, and the alternative of approximating it has until recently proven difficult.

We show how to estimate the parameters of virtually any diffusion model by simulated maximum likelihood (SML). The SML method works as follows. First, we construct consistent approximations to the transition densities of the diffusion and use these approximations to evaluate the likelihood function. Then, we maximize this approximated likelihood function. Since the approximations to the transition densities are consistent, so is the approximation to the likelihood function. This implies that asymptotically the SML estimator behaves just like the unattainable exact maximum likelihood estimator.