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Convex bounds on multiplicative processes, with applications to pricing in incomplete markets

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Abstract

Extremal distributions have been extensively used in the actuarial literature in order to derive bounds on functionals of the underlying risks, such as stop-loss premiums or ruin probabilities, for instance. In this paper, the idea is extended to a dynamic setting. Specifically, convex bounds on multiplicative processes are derived. Despite their relative simplicity, the extremal processes are shown to produce reasonably accurate bounds on option prices in the classical trinomial model for incomplete markets.

Keywords: Convex order; Extremal distributions; Incomplete market; Risk-neutral martingales; Option pricing; Trinomial model

1. Introduction

Stochastic orderings are probabilistic tools for comparing random variables or random vectors. Mathematically speaking, they are partial order relations defined on sets of probability distributions. Many papers have been devoted to the derivation of bounds in some stochastic order on a given random variable S. These bounds use some information about the random variable S, like moments, support, unimodality, etc. Relying on *extrema* with respect to some order relation, the actuary acts in a conservative way by basing his/her decisions on the least attractive risk that is consistent with the incomplete available information. The extrema correspond to the "worst" and the "best" risk. See, e.g., Denuit et al. (1999a) and the references therein.

The convex order will be extensively used in the present work. Recall that this stochastic order relation is defined as follows: given two random variables S and T, S is said to be smaller than T in the convex order, denoted as $S \preceq_{\text{cx}} T$, if the inequality $\mathbb{E}[\phi(S)] \leq \mathbb{E}[\phi(T)]$ holds for all the convex functions $\phi : \mathbb{R} \to \mathbb{R}$, provided the expectations exist. The intuitive meaning of $S \preceq_{\text{cx}} T$ is that S is less variable than T. The multivariate version of $\preceq_{\text{cx}}$ is easily obtained by considering convex functions $\phi : \mathbb{R}^n \to \mathbb{R}$. Another multivariate extension of $\preceq_{\text{cx}}$ consists in requiring $\preceq_{\text{cx}}$-inequalities among all the possible linear combinations of the components of the random vectors to be compared, as explained below.

In this paper, we consider multiplicative discrete-time processes $\{X_n, n = 1, 2, \ldots\}$ obtained as follows. Starting from a sequence $\{Y_n, n = 1, 2, \ldots\}$ of positive independent random variables, we define recursively the X_n's as

$$X_{n+1} = X_n Y_{n+1}, \quad n = 1, 2, \ldots \tag{1}$$

with $X_1 = Y_1$. Such a process can be seen as a multiplicative random walk with relative increase Y_n at time n. It is widely used in finance to model the price of financial instruments (where X_n is the exponential of some process with independent increments). Our aim is to derive lower and upper bounds on the process $\{X_n, n = 1, 2, \ldots\}$ in the sense that any positive linear combination of the X_n's is bounded in the convex order by the corresponding linear combinations of the components of the extremal processes. This is similar to the work by Koshevoy and Mosler (1998) where orderings between random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ defined by $a_1X_1 + a_2X_2 + \cdots + a_nX_n \preceq_{\text{cx}} a_1Y_1 + a_2Y_2 + \cdots + a_nY_n$ for all constants $a_1, a_2, \ldots, a_n$ are studied.

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The results derived in this paper are applied to discrete-time contingent claims pricing models. The underlying assets are assumed to follow a discrete-time process and trading only takes place at some prespecified dates. We consider an incomplete market framework, so that the risk-neutral probability measure is not unique and we are in presence of a class of risk-neutral measures. The aim is thus to find the risk-neutral probability measures that imply the lower and upper bounds on the price of the claim and that are elements of the class of admissible prices. Examples within a trinomial model (i.e. a model where the change in the value of the stock between two trading times can attain three different values) are discussed.

There is an obvious connection between the papers devoted to extremal distributions that appeared in the actuarial literature and financial pricing in incomplete markets. If there are no arbitrage opportunities then the financial pricing amounts to computing the expectation of the discounted payoff under a risk-neutral probability measure. In incomplete markets, there are infinitely many such risk-neutral measures. The class of risk-neutral probability measures can thus be considered as a class of distributions with fixed support and first moment. Extremal elements can then be identified within the set of risk-neutral distributions, leading to bounds on the prices of contingent claims. This bridge between actuarial risk theory and financial mathematics seems to be promising.

The paper is organized as follows. The extremal processes are built in Section 2. Section 3 describes the application to financial pricing in the trinomial model. Numerical illustrations are provided there. The final Section 4 concludes the paper with possible extensions to the results derived in the present work.

2. Extremal processes

2.1. Definitions

As explained in the introduction, we consider here discrete-time stochastic processes $\{X_n, n = 1, 2, \ldots\}$ built from a sequence of positive independent random variables $\{Y_n, n = 1, 2, \ldots\}$ by (1). If the random variable Y_i is valued in the interval $[a_i, b_i]$, let Y_i^- and Y_i^+ be two positive random variables such that $Y_i^- \preceq_{\rm cx} Y_i \preceq_{\rm cx} Y_i^+$ holds for all i. If $\mathbb{E}[Y_i] = \mu_i$ then $Y_i^- = \mu_i$ almost surely, and

$$Y_i^+ = \begin{cases} a_i \text{ with probability } \dfrac{b_i - \mu_i}{b_i - a_i}, \\ b_i \text{ with probability } \dfrac{\mu_i - a_i}{b_i - a_i}, \end{cases}$$

are known to be such that $Y_i^- \preceq_{\rm cx} Y_i \preceq_{\rm cx} Y_i^+$. Other choices for the $\preceq_{\rm cx}$-bounds are possible, according to the amount of information available about the Y_i's (support, moments, unimodality, ageing notions, etc.). See, e.g., Courtois and Denuit (2006) and the references therein.

All the random variables $Y_1, Y_2, \ldots, Y_1^-, Y_2^-, \ldots, Y_1^+, Y_2^+, \ldots$ are assumed to be independent. Starting from $X_1^- = Y_1^-$ and $X_1^+ = Y_1^+$, we define the extremal processes $\{X_n^-, n = 1, 2, \ldots\}$ and $\{X_n^+, n = 1, 2, \ldots\}$ by $X_i^- = X_{i-1}^- Y_i^-$ and $X_i^+ = X_{i-1}^+ Y_i^+$ for $i = 2, 3, \ldots$.

2.2. Convex ordered marginals

We expect that a convex ordering holds between X_i^-, X_i^+ and X_i. To prove that this is indeed the case, we will need the following useful lemma.

Lemma 2.1. *Let T_1, T_2, Z_1, Z_2 be independent and positive random variables such that $T_1 \preceq_{\rm cx} T_2$ and $Z_1 \preceq_{\rm cx} Z_2$. Then, $T_1 Z_1 \preceq_{\rm cx} T_2 Z_2$ holds.*

Proof. Let ϕ be a convex function, and let us denote as F_{T_1}, F_{T_2}, F_{Z_1} and F_{Z_2} the distribution functions of T_1, T_2, Z_1 and Z_2, respectively. From

$$\begin{aligned} \mathbb{E}[\phi(T_1 Z_1)] &= \int_0^\infty \mathbb{E}[\phi(t Z_1)] \mathrm{d}F_{T_1}(t) \\ &\le \int_0^\infty \mathbb{E}[\phi(t Z_2)] \mathrm{d}F_{T_1}(t) \quad \text{since } Z_1 \preceq_{\rm cx} Z_2 \\ &= \int_0^\infty \mathbb{E}[\phi(T_1 z)] \mathrm{d}F_{Z_2}(z) \\ &\le \int_0^\infty \mathbb{E}[\phi(T_2 z)] \mathrm{d}F_{Z_2}(z) \quad \text{since } T_1 \preceq_{\rm cx} T_2 \\ &= \mathbb{E}[\phi(T_2 Z_2)], \end{aligned}$$

we conclude that the announced $\preceq_{\rm cx}$-inequality indeed holds. □

We are now ready to prove the next result that shows that the processes $\{X_n^-, n = 1, 2, \ldots\}$, $\{X_n, n = 1, 2, \ldots\}$ and $\{X_n^+, n = 1, 2, \ldots\}$ have indeed $\preceq_{\rm cx}$-ordered univariate marginals.

Proposition 2.2. *The stochastic inequalities $X_i^- \preceq_{\rm cx} X_i \preceq_{\rm cx} X_i^+$ hold for all i.*

Proof. Let us prove the announced result using an iterative argument. The result is obviously true for $i = 1$, since it reduces to $Y_1^- \preceq_{\rm cx} Y_1 \preceq_{\rm cx} Y_1^+$. Now, assume that the result holds for $i = 1, 2, \ldots, n$ and let us prove it for $n + 1$. Let us apply Lemma 2.1 in our setting. Taking $T_1 = T_2 = X_n^-$ and $Z_1 = Y_{n+1}^-$, $Z_2 = Y_{n+1}$, we get

$$X_n^- Y_{n+1}^- = X_{n+1}^- \preceq_{\rm cx} X_n^- Y_{n+1}.$$

Now, taking $T_1 = T_2 = Y_{n+1}$ and $Z_1 = X_n^-$, $Z_2 = X_n$, we have

$$X_n^- Y_{n+1} \preceq_{\rm cx} X_n Y_{n+1} = X_{n+1}.$$

We then conclude that $X_{n+1}^- \preceq_{\rm cx} X_{n+1}$ by transitivity. The proof of $X_{n+1} \preceq_{\rm cx} X_{n+1}^+$ follows along the same lines. □

2.3. Convex ordered linear combinations

Let us now prove that any positive linear combination of the X_i's is bounded from below and from above in the $\preceq_{\rm cx}$-sense by the same combination of the X_i^-'s and of the X_i^+'s.