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Recursive equilibrium in stochastic OLG economies: Incomplete markets[☆]

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ABSTRACT

We prove generic existence of recursive equilibrium for overlapping generations economies with uncertainty and incomplete financial markets. Generic here means in a residual set of utilities and endowments. The result holds provided there is sufficient intragenerational household heterogeneity, and transition probabilities and the asset payoff matrix satisfy mild regularity conditions. The paper also provides a new methodological technique to establish comparative statics, or perturbation, properties in such environments.

1. Introduction

Recursive equilibria are the focus of much of the macroeconomic and financial applications of the OLG model with incomplete markets.[1] The focus on recursive equilibrium is due to its computational tractability, as well as to the epistemological appeal rational expectations have when forecasts are based on a simple state space. Yet, there is no proof of existence of such equilibria for these environments.

Equilibria fail to be recursive when current exogenous shocks and wealth distribution are not sufficient statistics of the evolution of economic variables. This is due to a multiplicity of equilibrium prices given these states: for example, it can well be that unique continuation equilibrium prices cannot be found unless past prices or multipliers are used to select among equilibria. Kubler and Polemarchakis (2004) indeed provided two examples of nonexistence of recursive equilibrium based on this multiplicity problem. Despite the fact that their examples seem knife-edge, it is not obvious to formally establish their nongeneric character, because of the infinite dimension of the space of histories equilibrium variables depend on. Even if one starts from simple Markov equilibria where the state space has already been conveniently reduced, an argument proving the one-to-one-ness of other endogenous variables as functions of exogenous shocks and wealth distribution appears to be nontrivial. This seemingly technical issue is at the core of the nonexistence problem and therefore needs to be addressed seriously.

In Citanna and Siconolfi (2010) we provided a framework to attack this problem. Basically, we took advantage of stationarity and the finite life span of individuals to study properties of their demand functions in a finite dimensional setting: for an individual born at any date-event, such demands are functions of the finitely many prices she will face during her lifetime. To decouple generations from each other, we discarded almost all the information that prices derive from the dynamic equilibrium restrictions (market clearing), and considered properties of these demand functions for all admissible prices. We then asked how demand changes as one of these lifetime prices changes. Failure of recursiveness implies that all such individuals simultaneously have the same age-one wealth even when facing different prices thereafter. The idea then was to show that with *enough intragenerational heterogeneity* in preferences and endowments there is almost always going to be an individual who will spend differently at different prices.

This framework was successfully applied in the simpler environment where financial markets are complete. When markets are sequentially complete, no arbitrage state prices and therefore individual multipliers are unique up to a normalization. As a consequence, generically some individuals at age one will have different multipliers when facing different prices. In turn, if multipliers

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1 Examples by now abound; see, e.g., Rios Rull (1996), Constantinides et al. (2002), Geanakoplos et al. (2004) and Storesletten et al. (2004). The general advantage of recursive formulations is apparent from the huge success that recursive methods have in the macroeconomic literature (for surveys, see e.g. Stokey and Lucas, 1989; Ljungqvist and Sargent, 2004).

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are different for some individual, then equality of wealth for all individuals is nongeneric: the cross-sectional wealth distribution is generically different.

With incomplete markets, state prices and therefore multipliers are not unique. Perturbing the economy may induce changes of multipliers at some history, possibly far away in the future, but their change may live in the orthogonal complement of the asset payoff matrix, so that it does not trickle down, that is, does not affect multipliers at histories closer to age one. The argument is therefore quite intricate and demands a substantial change in two aspects: (1) we need additional comparative statics conditions of individual demand that are specific to the incomplete market environment; (2) we need to weave these conditions together to guarantee that price changes trickle down. With these modifications in place, we show that there is a large (i.e., residual, dense) subset of economies parametrized by utilities where recursive equilibria exist. Moreover, their state space contains a generic set of wealth distributions that can be taken to be initial conditions for competitive equilibria of each economy considered. The result now requires some additional, albeit generic, assumptions. First, it must be possible to transfer positive wealth from today to tomorrow in each possible state. Second, assets must allow for arbitrary allocations of wealth across any chosen number of states, provided that this number is less than the number of assets (that is, the asset payoff matrix is in general position (see Section 2 for a definition)).
As with complete markets, generic existence is guaranteed provided there is also enough intragenerational heterogeneity in preferences and endowments. If H denotes the number of intracohort types, C the number of physical commodities, G the maximum after-birth number of cohorts, S the number of states of uncertainty, and I the number of financial assets traded in each period, this condition is here summarized[2] as

$$H > 2\left[(C-1)\sum_{a=0}^{G} S^a + I\sum_{a=0}^{G-1} S^a\right].$$

The paper also provides a new methodological insight into establishing comparative statics, or perturbation, properties of optimal solutions to the consumer problem under incomplete asset markets. Indeed, in order to track the effects of perturbations and to bypass the possibility that they may have no first order effect on financial wealth, and because of the spanning problem, we substantially extend the solution technique with respect to Citanna and Siconolfi (2010). This mathematical structure now takes center stage of the perturbation analysis, and is used to guide the reader through the perturbation argument.

Our class of OLG economies has multiple goods, generations and types within each generation. We present the model in its simplest form, where there are only short-lived real – namely, numéraire – assets in zero net supply. Long-lived assets, e.g., with nonzero real payoffs and in positive net supply, as well as production (as in Rios Rull, 1996) can be easily added without altering the substance of the proofs, but complicating the already heavy notation. Hence, they are only briefly discussed in Section 5. The discussion of economies with idiosyncratic risks is unaltered by the structure of financial markets, and the conclusions are the same as in Citanna and Siconolfi (2010), and worth recalling here: if individual risks are considered, such as survival or unemployment risks, the degree of intracohort heterogeneity required for the argument can be substantially reduced. The left-hand side of the inequality above now reads $H\Sigma$, where H are the number of ex-ante different types of individuals, and Σ is the cardinality of the individual risk space. In particular, our results can go through even if $H = 1$, i.e., even if there is no ex-ante heterogeneity in household characteristics, an assumption often made in applications.

As with many other modeling solutions in the applied sciences, the successful application of the notion of recursive equilibria involves mathematical as well as numerical foundations. Our proof is not constructive (i.e., an algorithm), and lacks a method to verify, in practice, whether an economy is or not part of the generic set. However, it implies that *for all practical purposes* such an equilibrium notion exists thereby providing sound foundations to computational methods. That is, all but a negligible subset of parameters have recursive equilibria. Applied macroeconomics uses a low number H of types and parametric utility functions as such these economies lie in lower dimensional set of the parameter space that we consider. Nevertheless, our result does not imply that such economies do not have a recursive equilibrium, rather that if this were the case, the nearby economy with recursive equilibria *may* lie outside this lower dimensional set.

2. The model

We consider standard stationary OLG economies. Time is discrete, indexed by $t = 0, 1, 2, \dots$. There are $S > 1$ states of the world[3] that may realize in each period. A *history* up to and including date t is an array of states, one for each date $\tau \le t$, and denote it by $s^t = (s_0, s_1, \dots, s_t)$; we also write $s^t = (s^{t-1}, s_t)$ and $s^t = (s_0, s^{t-1})$ when convenient. We denote with $S^t_{s_0}$ the set of all possible histories of length t with initial shock s_0. We say that history s^t precedes history $\bar{s}^{t'}$ and we write $\bar{s}^{t'} \succsim s^t$ if there is an array of $t'-t$ realizations of states $s^{t'-t}$ such that $\bar{s}^{t'} = (s^t, s^{t'-t})$. Then, $\widetilde{S}_{s_0} = \cup_t S^t_{s_0}$ together with $\succsim$ is a tree with nodes s^t and root s_0. When all possible initial conditions need to be considered at once, the union $\tilde{S} = \cup_{s_0\in S} \widetilde{S}_{s_0}$ of S trees with distinct roots comes into play. All the variables introduced below, including state of the world realizations, are seen as stochastic processes adapted to $\tilde{S}$, and their realization at time t is denoted with subscript t. Also, let $\mathbb{E}_t$ denote the conditional expectation at t.

At each t, H individuals enter the economy. Each individual $h \in H$ lives $G + 1 \ge 2$ periods, indexed by $a = 0, \dots, G$, from the youngest ($a = 0$) to the oldest ($a = G$) age. If and when we need to make explicit that a variable or function is of a specific individual of type h (and of age a), we use the superscript h (and ha). At each t, $C \ge 1$ physical commodities are available for consumption. The consumption bundle of an individual h of age a is $x^{ha}_t \in \mathbb{R}^C_{++}$, with $x^{ha}_{c,t}$ denoting the consumption of commodity c at t.

Each individual h has a time- and state-separable expected utility function, with state and age dependent Bernoulli utility index u^{ha}_t at age a and time t. At each t, endowments are e^{ha}_t for $h \in H$ and $0 \le a \le G$.

Bernoulli utility indexes u^{ha}_t and endowments e^{ha}_t satisfy:

- $e^{ha}_s \in \mathbb{R}^C_{++}$, for all $h \in H$, all a and all s; endowment profiles live in the interior of the positive cone of dimension $H(G+1)S$;
- $u^{ha}(\cdot, s) : \mathbb{R}^C_{++} \to \mathbb{R}$ is twice continuously differentiable, differentially strictly increasing, differentially strictly concave (the Hessian is negative definite), and satisfies the boundary condition: if $x^{ha}_c \to 0$, then $\left\| Du^{ha}(x^{ha}, s)\right\| \to +\infty$, where Du^{ha} denotes the vector of partial derivatives.

[2] Throughout the paper, a superscript, e.g., S^a, denotes power when applied to capital letters, while it is an index when applied to small and to calligraphic letters.

[3] As standard practice in general equilibrium, capital letters are used to denote both a set and its cardinality.