Journal of Mathematical Economics 48 (2012) 437–444

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Failure of the index theorem in an incomplete market economy

Takeshi Momi *

Department of Economics, Doshisha University, Kamigyo-ku, Kyoto 602-8580, Japan

ARTICLE INFO

Article history:
Received 6 April 2011
Received in revised form
28 February 2012
Accepted 30 July 2012
Available online 10 September 2012

Keywords:
Incomplete market
Index theorem
Homotopy

ABSTRACT

Recently, it was proved that the index of an economy with incomplete real asset markets is typically +1 when the degree of incompleteness, which is defined as the difference between the number of states and the number of securities, is an even number. This paper considers the case where the degree of incompleteness is an odd number and proves that any odd number can be realized as the index of such an economy.

1. Introduction

The index theorem was introduced to economics by Dierker (1972), who applied it to the Arrow–Debreu exchange economy. A unique feature of the index theorem is its detective power with respect to multiple equilibria. The index theorem insists that the sum of the indices at each equilibrium over all the equilibria of an economy, that is, the index of the economy, equals +1. Therefore, the existence of an equilibrium with index −1 implies that the economy has at least two other equilibria with index +1.

The general equilibrium model with incomplete markets is an extension of the Arrow–Debreu model in that it describes the trading mechanism for uncertainty in a more precise manner. A natural question to follow is whether or not the index theorem holds for the incomplete market model. Recent works by Momi (2003), Bich (2006), and Predtetchinski (2006) gave a partial answer to this question: the index theorem typically holds when the degree of incompleteness, which is defined as the difference between the number of states and the number of securities, is an even number. The purpose of this paper is to study the case where the degree of incompleteness is an odd number.

This paper shows that any odd number can be the index of an incomplete market economy in which the degree of incompleteness is odd. That is, the index theorem does not hold for such an incomplete market economy. This is in sharp contrast to Dierker's index theorem for the Arrow–Debreu economy.

We show the existence of an economy whose index is not equal to +1 by conversely following the arguments of Momi (2003). First, we draw a picture of paths such that an arbitrarily given odd number is induced as the index of the picture when Momi's index change rule is applied. Next, we find an aggregate excess demand function that realizes the drawn paths as its homotopy paths. Finally, we construct an economy that realizes the aggregate excess demand function by adapting the result by Momi (2010), which shows the characterization of aggregate excess demand functions in incomplete markets.

The paper is organized as follows. Section 2 describes the setup of the incomplete market model. Section 3 states the main result of the paper. Section 4 reviews the approach in Momi (2003), on which this paper is based. Section 5 provides the proof of the main result.

2. The incomplete market model

We consider a standard two-period economy with incomplete real asset markets. There are S possible states in the second period and N goods in each state so that R^M, where $M = (S+1)N$ with period 0 as state 0, represents the total commodity space. There are $J(\leq S)$ real assets $A^j, j = 1, \ldots, J$, each of which promises the delivery of a bundle of commodities $A^j_s = (A^j_{s1}, \ldots, A^j_{sN})$ if state $s \in \{1, \ldots, S\}$ occurs in the second period. We represent the asset structure by $A = \{A^j_{sn}\}$ and assume that the assets are unredundant. Each consumer indexed by i is defined by $(\precsim^i, \omega^i)$ where $\precsim^i$ is the consumer's strictly convex, monotonic, continuous, and complete preference ordering on the consumption set R^M_+ and $\omega^i \in R^M_{++}$ is the consumer's initial endowment vector. Let p, x^i, ω^i, and $z^i = x^i - \omega^i$ respectively denote price, consumption, endowment, and excess demand of consumer i, where $p = (p_0 \ldots, p_S)$ and $p_s = (p_{s1}, \ldots, p_{sN})$, and so on. We often relabel p

* Tel.: +81 75 251 3647.
E-mail address: tmomi@mail.doshisha.ac.jp.

0304-4068/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.jmateco.2012.07.004

as $p = (p_1, \ldots, p_l, \ldots, p_M)$, where $p_1 = p_{01}, p_2 = p_{02}, \ldots, p_M = p_{SN}$ and relabel x^i, ω^i, and z^i in a similar manner.

The budget set, which the excess demand vector $z = (z_0, \ldots, z_S) \in R^M$ satisfies, is

$$L(p) = \left\{ z \in R^M \,\middle|\, \begin{array}{l} pz = 0, \\ \mathbf{p_1} \Box \mathbf{z_1} \in \langle A(p) \rangle \end{array} \right\},$$

where $p = (p_0, \ldots, p_S) \in \Delta = \{p \in R^M_{++} | \|p\| = 1\}$ is the present value price system, $\mathbf{p_1} \Box \mathbf{z_1}$ denotes $[p_1 z_1, \ldots, p_S z_S]^T$, and $\langle A(p) \rangle$ denotes the linear space in R^S spanned by J column vectors of the $S \times J$ payoff matrix $A(p) \equiv (p_s A^j_s)^{j=1,\ldots,J}_{s=1,\ldots,S}$. See Duffie and Shafer (1985) for the justification behind this definition of a budget set using the present value price system. The excess demand function of consumer i is thus defined by

$$z^i(p) = \big\{ x - \omega^i \in R^M | (x - \omega^i) \in L(p) \text{ and } x' \precsim^i x \text{ for any } x' \in R^M_+ \text{ satisfying } (x' - \omega^i) \in L(p) \big\}.$$

The equilibrium price $\bar{p}$ is defined by $\sum_i z^i(\bar{p}) = 0$. Thus, our incomplete market economy is defined by the set of consumers of a finite number and an asset structure $(\{\precsim^i, \omega^i\}_i, A)$.

The budget set $L(p)$ is a $k \equiv M - (S - J) - 1$-dimensional linear subspace in R^M if and only if the payoff matrix $A(p)$ is of full column rank. We label such a price as "good" and write Δ^g to denote the set of good prices $\Delta^g = \{p \in \Delta | \mathrm{rank} A(p) = J\}$. We also label the critical price where the rank of $A(p)$ is less than J (that is, where the dimension of $L(p)$ is less than k) as "bad" and define $\Delta^b = \Delta \setminus \Delta^g = \{p \in \Delta | \mathrm{rank} A(p) < J\}$. Since the budget set $L(p)$ drops its dimension at bad prices, the excess demand $z^i(p)$ is not continuous at bad prices.

We consider the first consumer $i = 1$ as unconstrained and write her excess demand function as

$$z^u(p) = \big\{ x - \omega^1 \in R^M | p(x - \omega^1) = 0 \text{ and } x' \precsim^i x \text{ for any } x' \in R^M_+ \text{ satisfying } p(x' - \omega^1) = 0 \big\}$$

to distinguish it from her constrained excess demand $z^1(p)$. Let $Z^c = \sum_{i \geq 2} z^i$ denote the aggregate excess demand function of the remaining constrained consumers $i \geq 2$ and let $Z = z^u + Z^c$ denote the aggregate excess demand of the economy. See Duffie and Shafer (1985) for the justification of considering an unconstrained consumer. In particular, $Z(\bar{p}) = 0$ implies $\sum_i z^i(\bar{p}) = 0$, and $\sum_i z^i(\bar{p}) = 0$ implies the existence of p' such that $Z(p') = 0$. Further, because of Walras' law, one of the M equations of $Z(p) = 0$ is redundant, and hence $\hat{Z}(\bar{p}) = 0$, where the symbol "ˆ" indicates that the M-th element is dropped, implies that $\bar{p}$ is an equilibrium price.

Moreover, it is sometimes convenient to consider the excess demand defined for k-dimensional linear subspaces of R^M, because $L(p)$ is such a space when $p \in \Delta^g$. We let $G^k(R^M)$ denote the set of k-dimensional linear subspaces of R^M, called the Grassmann manifold. We define $G^k_{++}(R^M) = \{L \in G^k(R^M) | L \bigcap R^M_+ = 0\}$. For a good price $p \in \Delta^g$, $L(p)$ is an element of $G^k_{++}(R^M)$. We define

$$\tilde{z}^i(L) = \big\{ x - \omega^i \in R^M | (x - \omega^i) \in L \text{ and } x' \precsim^i x \text{ for any } x' \in R^M_+ \text{ satisfying } (x' - \omega^i) \in L \big\}$$

for $L \in G^k_{++}(R^M)$, and define $\tilde{Z}^c(L) = \sum_{i \geq 2} \tilde{z}^i$. Clearly, $\tilde{z}^i(L(p)) = z^i(p)$ for $p \in \Delta^g$.

As mentioned above, the function $p \mapsto L(p)$ is not continuous at $p \in \Delta^b$. To avoid this discontinuity, we consider a pseudo-equilibrium. Remember that $\langle A(p) \rangle$ is a J-dimensional linear subspace in R^S for $p \in \Delta^g$. For $p \in \Delta$ and $g \in G^J(R^S)$ where $G^J(R^S)$ denote the set of J-dimensional linear subspaces of R^S, we let $\mathcal{L}(p, g)$ denote the budget set

$$\mathcal{L}(p, g) = \left\{ z \in R^M \,\middle|\, \begin{array}{l} pz = 0, \\ \mathbf{p_1} \Box \mathbf{z_1} \in g \end{array} \right\}.$$

Note that $\mathcal{L}(p, g)$ is always an element of $G^k_{++}(R^M)$ and the function $\mathcal{L} : \Delta \times G^J(R^S) \to G^k_{++}(R^M)$ is continuous. The pair $(\bar{p}, \bar{g}) \in \Delta \times G^J(R^S)$ is called a pseudo-equilibrium when it satisfies $z^u(\bar{p}) + \tilde{Z}^c(\mathcal{L}(\bar{p}, \bar{g})) = 0$ and $\langle A(\bar{p}) \rangle \subset \bar{g}$. If $(\bar{p}, \bar{g})$ is a pseudo-equilibrium and $A(\bar{p})$ is of full column rank, then $\langle A(\bar{p}) \rangle = \bar{g}$ and $\bar{p}$ is an equilibrium price.

We often represent $g \in G^J(R^S)$ by local coordinate systems as in Momi (2003). We let Σ denote the set of permutations of $\{1, \ldots, S\}$ and Π_σ be the permutation matrix such that $\Pi_\sigma \begin{bmatrix} X_1 \\ \vdots \\ X_S \end{bmatrix} = \begin{bmatrix} X_{\sigma(1)} \\ \vdots \\ X_{\sigma(S)} \end{bmatrix}$ for each $\sigma \in \Sigma$. By some $\sigma \in \Sigma$ and $Y = \begin{bmatrix} Y_{11} & \cdots & Y_{1J} \\ \vdots & & \vdots \\ Y_{(S-J)1} & \cdots & Y_{(S-J)J} \end{bmatrix} \in R^{(S-J)J}$, any $g \in G^J(R^S)$ can be written as

$$g = \{ w \in R^S | [I | Y] \Pi_\sigma w = 0 \},$$

where I denotes the $(S-J) \times (S-J)$ identity matrix. For each $\sigma \in \Sigma$, define $W_\sigma = \{g \in G^J(R^S) |$ there is $Y \in R^{(S-J)J}$ such that $g = \{w \in R^S | [I|Y]\Pi_\sigma w = 0\}\}$ and define $\varphi_\sigma : W_\sigma \to R^{(S-J)J}$ by $g = \{w \in R^S | [I | \varphi_\sigma(g)] \Pi_\sigma w = 0\}$. Then, $\{W_\sigma, \varphi_\sigma\}_{\sigma \in \Sigma}$ is an atlas for the Grassmann manifold $G^J(R^S)$.

3. Result

We define the index at an equilibrium price $\bar{p} \in \hat{Z}^{-1}(0)$ by

$$\mathrm{index}\, \hat{Z}(\bar{p}) = (-1)^{M-1} \mathrm{sign}\left(\det\left[\frac{\partial \hat{Z}}{\partial \hat{p}}(\bar{p}) \right] \right),$$

where $\left[\frac{\partial \hat{Z}}{\partial \hat{p}}\right]$ is the Jacobian matrix of Z whose last row and last column are dropped.[1] The index of the economy is defined by $\sum_{\bar{p} \in \hat{Z}^{-1}(0)} \mathrm{index}\, \hat{Z}(\bar{p})$. This definition is equivalent to that in Momi (2003) regardless of the different price normalizations and is a natural extension of the index of the Arrow–Debreu economy. See Momi (2003) for the proof that the definition of the index is independent from price normalizations. The index of an economy is well defined only when $\hat{Z}^{-1}(0)$ is a finite set and the derivative of $\hat{Z}$ is well defined at every $\bar{p} \in \hat{Z}^{-1}(0)$.

Theorem. *Let α be an odd number. There exists an economy $(\{\precsim^i, \omega^i\}_i, A)$ with some finite numbers of states S, assets J, commodities N and consumers, whose index is α.*

4. Sketch of proof

This section briefly reviews Momi's (2003) approach and sketches the proof of the theorem. As in Brown et al. (1996), we define the homotopy $H : \Delta^g \times [0, 1] \to R^M$ as

$$H(p, t) = z^u(p) + tZ^c(p).$$

Although H is defined on $\Delta^g \times [0, 1]$, where H is continuous, we often work with $\overline{H^{-1}(0)}$, the closure of $H^{-1}(0)$ in $\Delta \times [0, 1]$.

When each preference ordering $\precsim^i$ is represented by a smooth utility function, for almost all initial endowments $\omega = \{\omega^i\}_i$ and asset structures A, $H^{-1}(0)$ is typically drawn as in Fig. 1. We call a one-dimensional connected manifold as a "path" and we call a path in $H^{-1}(0)$ or in $\overline{H^{-1}(0)}$ as a homotopy path. We call each point

[1] Note that $\mathrm{sign}(a)$ denotes the sign of $a \in R$: $\mathrm{sign}(a) = +1, 0, -1$ if $a > 0, a = 0, a < 0$, respectively; $\det[X]$ denotes the determinant of matrix $[X]$.