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ScienceDirect
Journal of Mathematical Economics 42 (2006) 626–635
JOURNAL OF Mathematical ECONOMICS
www.elsevier.com/locate/jmateco

A new proof of the index formula for incomplete markets

Arkadi Predtetchinski*

Maastricht University, Faculty of Economics and Business Administration, Tongersestraat 53, 6200 MD Maastricht, Limburg, Netherlands

Received 15 March 2006; accepted 6 April 2006
Available online 23 June 2006

Abstract

This paper gives a new proof of the index formula established by [Momi, T., 2003. The index theorem for a GEI economy when the degree of incompleteness is even. Journal of Mathematical Economics 39, 273–297] for an economy with incomplete asset markets where the difference between the number of states (S) and the number of assets (J) is an even number. The proof uses a single globally defined homotopy function on the asset pseudo-equilibrium manifold connecting the excess demand of a given economy to the individual excess demand of the unconstrained agent. We show that the asset pseudo-equilibrium manifold is orientable if the number $S - J$ is even and deduce the index formula from the homotopy invariance theorem for the degree of a map.

JEL classification: D52

Keywords: Incomplete markets; Index formula; Homotopy

1. Introduction

The index theorem was first introduced into economics by Dierker (1972). The theorem states that the indices of the individual equilibria in a regular Arrow–Debreu economy add up to $+1$. Recently Momi (2003) has proved the index formula for asset market economies where the difference between the number of states (S) and the number of available assets (J) is an even number.

* Tel.: +31 43 388 3906; fax: +31 43 388 4878.
E-mail address: A.Predtetchinski@algec.unimaas.nl (A. Predtetchinski).

0304-4068/$ – see front matter
doi:10.1016/j.jmateco.2006.04.002

The essential difficulty involved in the proof of the index formula for incomplete markets is the discontinuity of the excess demand function, caused by changes in the rank of the asset return matrix, as prices vary. While for the Arrow–Debreu economy the index formula is implied by the fact that the aggregate excess demand function is homotopic to the individual excess demand function of a single agent via a proper homotopy map, Momi (2003) has to rely on the system of switching homotopies introduced by Brown et al. (1996) and on the system of local homotopy functions of Demarzo and Eaves (1996).

This paper presents a new proof of the result by Momi (2003). We introduce a single globally defined homotopy function between the excess demand of a given economy and the excess demand of an unconstrained agent. The domain of the homotopy function is the so-called asset pseudo-equilibrium manifold as introduced in Zhou (1997). Assuming that the number $S - J$ is even, we prove the asset pseudo-equilibrium manifold to be orientable and derive the index formula from the homotopy-invariance property of the degree of a map.

The rest of the paper is organized as follows. In Section 2 the economy with incomplete asset markets is presented. In Section 3 the index formula is stated and some motivation for the new proof of this result is provided. Section 4 discusses the mathematical concepts used in the proof of the index formula. In Section 5 we show the asset pseudo-equilibrium manifold to be an orientable manifold, provided that the number $S - J$ is even. Section 6 completes the proof of the index formula.

2. The economy

We consider two-period economies with uncertainty represented by a finite set $\{1, \ldots, S\}$ of states of nature. There are L goods in period 0 and L goods in each state of nature in period 1. The total number of time and state-contingent commodities in the economy is therefore $M = (S + 1)L$. There are I agents in the economy, agent i characterized by a utility function $u^i : \mathbb{R}^M_{++} \to \mathbb{R}$ and a vector of initial endowments $e^i \in \mathbb{R}^M_{++}$. In addition, there are J assets in the economy characterized by an $(SL \times J)$-dimensional matrix A of payoffs. The entry a^j_{sl} of the matrix A specifies the amount of commodity l paid by asset j in the state of nature s.

We impose the following assumptions.

(A1). Functions u^i are twice continuously differentiable.

(A2). For each $x^i \in \mathbb{R}^M_{++}$ the vector $\mathrm{d}u^i(x^i)$ of partial derivatives of u^i at x^i belongs to $\mathbb{R}^M_{++}$.

(A3). For each $\bar{x}^i \in \mathbb{R}^M_{++}$ the closure of the set $\{x^i \in \mathbb{R}^M_{++} | u^i(x^i) \geq u^i(\bar{x}^i)\}$ is contained in $\mathbb{R}^M_{++}$.

(A4). If $x^i \in \mathbb{R}^M_{++}$ and $h \in \mathbb{R}^M \backslash \{0\}$ are such that $\mathrm{d}u^i(x^i)h = 0$ then $h^\top \mathrm{d}^2 u^i(x^i) h < 0$.

Assumptions (A1)–(A4) are standard in the theory of incomplete markets. In particular, this set of assumptions is employed in Duffie and Shafer (1985) to demonstrate generic existence of GEI-equilibrium.

We introduce some notation. Given natural numbers N and K we write $M(N, K)$ to denote the set of all $(N \times K)$-dimensional matrices. The symbol $M_j(N, K)$ denotes the subset of $(N \times K)$-matrices having rank j. For $D \in M(N, K)$ we write $\mathrm{span} D$ to denote the linear space spanned by the columns of D and D^k to denote the k^{th} column of the matrix D. Given two M-dimensional vectors $p = (p_{sl})$ and $z = (p_{sl})$ where $s = 0, 1, \ldots, S$ and $l = 1, \ldots, L$, the symbol $p \Box z$ denotes an S-dimensional vector with components $\sum_{l=1}^{L} p_{sl} z_{sl}$ for $s = 1, \ldots, S$. Given a matrix $A \in M(SL, J)$