Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER Journal of Economic Dynamics & Control 31 (2007) 3591–3613
JOURNAL OF Economic Dynamics & Control
www.elsevier.com/locate/jedc

A computational scheme for the optimal strategy in an incomplete market

Jussi Keppo[a], Xu Meng[a], Michael G. Sullivan[b,*]

[a] *Department of Industrial and Operations Engineering, University of Michigan, USA*
[b] *Department of Mathematics and Statistics, University of Massachusetts, Lederle Graduate Research Tower, Amherst, MA 01003-9305, USA*

Received 5 April 2005; accepted 20 December 2006
Available online 20 February 2007

Abstract

We examine the optimal portfolio selection problem of a single agent who receives an unhedgeable endowment. The agent wishes to optimize his/her log-utility derived from his/her terminal wealth. We do not solve this problem analytically but construct a recursive computational algorithm which approximates the optimal one. We present an 'intelligent' initial portfolio which requires, numerically, about 25% fewer corrective steps in the algorithm than a random initial portfolio, and outperforms the portfolio which ignores the unhedgeable risk of the endowment.

JEL classification: D52; G11

Keywords: Utility maximization; Incomplete markets; Endowment uncertainty; Numerical methods

1. Introduction

A market is incomplete if the uncertainties in the economy cannot be hedged by trading the market's financial instruments. An example of this uncertainty is when

*Corresponding author. Tel.: +1 413 545 1909.
E-mail address: sullivan@math.umass.edu (M.G. Sullivan).

0165-1889/$ - see front matter
doi:10.1016/j.jedc.2006.12.006

the agent cannot hedge the uncertainties of his/her endowment process.[1] In such a situation, we introduce a numerical technique to approximate the optimal trading strategy for an agent who wishes to optimize his/her log-utility from terminal wealth.

We extend the initial market to a complete market, where the endowment process can be hedged by creating a fictitious risky asset. By introducing a sequence of completed markets, we numerically find an optimal trading strategy which calls for an arbitrarily small investment in the fictitious asset. Since redistributing this small allocation among the real assets hardly affects the agent's terminal wealth, this modified strategy achieves an expected utility which is close to the best that the agent can achieve in the true market. In this sense, we approximate the true optimal strategy.

The idea of 'completing' the market begins with Karatzas et al. (1991). Using the same martingale method, Cvitanic and Karatzas (1996) extend the result to more general situations. Cuoco (1997) studies a similar problem but with an incomplete endowment. Instead of equating the problem to a 'dual' minimization problem, as done in Karatzas et al. (1991) and Cvitanic and Karatzas (1996), Cuoco (1997) proves that a solution exists for the original maximization problem. Cvitanic et al. (1999), Kramkov and Schachermeyer (1999), Karatzas and Zitkovic (2003), and Zitkovic (2005) solve the optimization problem in incomplete markets where the assets follow a more general semi-martingale model. These papers are similar in that they prove the existence of an optimal strategy, normally as the limit of some sequence.

Analytically deriving a sequence of trading strategies which converges to the optimal strategy is hard, even if we specialize to the relatively tractable case of log-utility and geometric Brownian motion. This difficulty is because the volatility of the budget equation that the optimal portfolio should hedge depends on the portfolio process itself. This leads to a 'stochastic fixed point problem' and, as proved, e.g., in Cvitanic et al. (1999), the optimal portfolio is guaranteed to exists only in a larger space, where the martingale method does not necessary hold. In the present paper we solve the problem by creating numerically a sequence of completed markets whose optimal trading strategies approximate the unique optimal strategy.

As a first step in this sequence, we alter the problem slightly so that the optimal strategy in the new problem is obtained from a solvable fixed point problem, leading to a so-called 'myopic' optimal portfolio strategy. The myopic strategy is suboptimal for the original problem, but we use it as the first step in our recursive algorithm. The myopic strategy is a good initial strategy in this algorithm because fewer steps are needed to converge to the optimal strategy compared to starting with some random initial strategy.

We show that the optimal and myopic portfolio process are better than the portfolio for an agent who ignores the endowment risk he/she cannot hedge. For instance, if the expected return of a risky asset is higher than the risk-free rate, the

[1]For instance, an energy company whose supply risk cannot be hedge in financial markets (e.g. Keppo, 2002), and an investor whose labor income does not perfectly correlate with financial assets (e.g., Duffie et al., 1997).