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Attainability of European path-independent claims in incomplete markets

Nicole Branger, Angelika Esser, Christian Schlag *

Faculty of Economics and Business Administration, Goethe University Frankfurt, Mertonstr. 17, Uni-Pf 77, D-60054 Frankfurt am Main, Germany

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Abstract

In this paper we consider the question which path-independent claims are attainable through self-financing trading strategies in an incomplete market. For continuous-time stochastic volatility models we show that only affine payoffs can be replicated. We provide a simple proof for this proposition based on the requirement that, for replication, the stock and the claim must be locally perfectly correlated, and based on the partial differential equation that any path-independent claim has to satisfy. Moreover, we show that this result does not carry over to discrete setups.

Keywords: Incomplete markets; Attainability; Stochastic volatility; Superhedging

1. Introduction and motivation

A market is called incomplete, if there exist payoff patterns that cannot be replicated by self-financing dynamic trading strategies. An immediate consequence of market incompleteness is that not all derivative assets can be valued using the standard pricing techniques based on replication. For example, it is well known that in an economy where the stock price exhibits stochastic volatility (SV) or jumps of random sizes it is impossible to uniquely price a European call option using only no arbitrage considerations. These results raise the fundamental question which types of path-independent payoffs are

* Corresponding author.
E-mail addresses: branger@wiwi.uni-frankfurt.de (N. Branger), Angelika.Esser@dekabank.de (A. Esser), schlag@wiwi.uni-frankfurt.de (C. Schlag).

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attainable at all. It is clear that European-type payoffs which are affine in the terminal stock price can always be generated by a simple buy-and-hold strategy in the stock and a default-free discount bond. If interest rates are stochastic but no bonds are traded, like in the short rate model of Cox et al. (1985), the set of trivially attainable path-independent claims reduces even further to those payoffs that are linear in the terminal stock price. Then, although it is clear that *not all* payoffs are attainable in an incomplete market, one would intuitively think that due to the huge set of trading strategies it should be possible to generate at least *some* path-independent payoffs which are not affine in the terminal stock price. However, we will prove as our main result in this paper that the set of these additional attainable path-independent payoffs is empty for a wide class of commonly used continuous-time option pricing models. We derive this result for models with SV, like those introduced by Heston (1993), Schöbel and Zhu (1999), or Bakshi et al. (1997).

Models with discrete time and discrete state variables are often used as simple but powerful analogies to continuous-time models. Concerning the issues analyzed in this paper, however, we will show that the analogies between the two classes of models are limited. Whereas it is impossible to generate nonlinear path-independent payoffs via self-financing trading strategies in the familiar continuous-time models, one can construct a path-independent payoff that is a nonlinear function of the terminal stock price in discrete models, but still replicable. The intuitive reason for this result is that in a discrete model the number and range of states that can be reached at the end of the period is always finite, so that we can construct 'locally linear' payoff patterns which in some cases aggregate to a nonlinear terminal cash flow pattern. However, in a discrete framework we must check every payoff pattern individually for its attainability.

In Section 2 we describe the diffusion setup with stochastic volatility. We then derive the main results concerning the attainability of path-independent claims in continuous-time models with incomplete markets. By means of a counterexample, we demonstrate that this property of continuous-time models does not carry over to discrete models, constituting an important discrepancy between the two types of models. Section 3 contains some concluding remarks.

2. Path-independent claims on incomplete markets

2.1. Stochastic volatility model in continuous time

Consider the SV model given by the stochastic differential equations

$$dS_t = \mu_S(t, S_t, V_t) S_t \, dt + V_t S_t \, dW_t^S, \tag{1}$$

$$dV_t = \mu_V(t, S_t, V_t) \, dt + \sigma_V(t, S_t, V_t) \, dW_t^V, \tag{2}$$

where dW_t^S and dW_t^V are correlated with correlation coefficient $\rho \in (-1, +1)$ and $\sigma_V \neq 0$. Volatility is assumed to be non-traded, so that the market is incomplete. The restriction on ρ is important, since in the degenerate case $|\rho| = 1$ the market would again be complete with only one source of risk. The market price of volatility risk, λ_V, is assumed to be a function of the current stock price, the current level of volatility V_t, and time, i.e.,