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Strategic behavior in non-atomic games[☆]

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ABSTRACT

In order to remedy the possible loss of strategic interaction in non-atomic games with a societal choice, this study proposes a refinement of Nash equilibrium, *strategic* equilibrium. Given a non-atomic game, its perturbed game is one in which every player believes that he alone has a small, but positive, impact on the societal choice; and a distribution is a *strategic* equilibrium if it is a limit point of a sequence of Nash equilibrium distributions of games in which each player's belief about his impact on the societal choice goes to zero. After proving the existence of strategic equilibria, we show that all of them must be Nash. We also show that all regular equilibria of smooth non-atomic games are strategic. Moreover, it is displayed that in many economic applications, the set of strategic equilibria coincides with that of Nash equilibria of large finite games.

1. Introduction

Modeling economic situations featuring a large number of agents with non-atomic games is especially convenient because the inability of players to affect societal variables provides significant technical ease. However, this advantageous feature may result in the dismissal of the strategic behavior desired to be depicted. Although admittedly extreme, the following example delivers a clear portrait of this point: Consider a game where players' choices have to be in $\{0, 1\}$, and their payoffs depend only on the average choice. Because that a player's action does not affect the average choice and, thus, his own payoff, any player is indifferent between any of his choices, and as a result any strategy profile is a Nash equilibrium. On the other hand, the unique plausible Nash equilibrium is one where each player chooses the highest integer, because this strategy is the unique Nash equilibrium of the finite, but arbitrarily large, player version of the same game.

Such failure of (lower hemi) continuity of the equilibrium correspondence in non-atomic games casts some doubts on the usefulness of the continuum model. Indeed, Aumann (1964) regarded it as a mathematically convenient approximation to the "true" model featuring a finite number of players. But, unlike the non-atomic model in Aumann (1964) which provides a clean solution to the core-equivalence problem that would work only in an approximate way in finite models, the above example shows that, in some games, the continuum model is not a good approximation to the finite one. Further examples are given in Novshek and Sonnenschein (1983).

Naturally, this issue has been widely investigated and several reassuring results have been obtained (see, among many others, Hildenbrand (1974), Postlewaite and Schmeidler (1978) and Mas-Colell (1983)). However, for the class of games we consider, in general, Nash equilibria of non-atomic games correspond to limit points of approximate equilibria of sequences of finite-player games converging to the original (see Carmona and Podczeck, 2011). In an approximate equilibrium the action played by each one of a large fraction of players must yield a payoff close to the maximum he or she can achieve. And, in general, it is not possible to obtain a similar result using exact equilibria of the approximating large finite games even for regular equilibria. We show this using a notion of regular equilibria analogous to those of Harsanyi (1973) and van Damme (1991).

Given the above difficulties, the current paper proposes a refinement of Nash equilibrium in non-atomic games, *strategic equilibrium* (henceforth to be abbreviated by SE), designed to alleviate these problems in a tractable way. In fact, our goal is

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to develop an equilibrium concept for non-atomic games that intuitively has the same properties of the limit points of equilibria of large finite games (the precise meaning of this will be illustrated below) and, at the same time, its existence is generally guaranteed. Furthermore, the identification of SE is relatively easier compared with that of limit points of equilibria of large finite games (henceforth, limit equilibria). In other words, as in Aumann (1964), we want to keep the analytical convenience of non-atomic games and, at the same time, to focus on equilibria of non-atomic games that provide a more accurate approximation to the equilibria of finite-player versions of these games. Perhaps more importantly, we show that in non-atomic games with finitely many actions and payoff functions, the latter being sufficiently smooth (such a game is henceforth referred to as a smooth game), every regular equilibrium is a SE. In this light, SE can be regarded as an extension of regular equilibrium for general non-atomic games.

This study presents and analyzes the concept of SE for non-atomic games in which the payoff of each agent depends on what he chooses and on the distribution of actions chosen by the other players (which we refer to as the societal choice). For any non-atomic game and $\varepsilon > 0$, we define an ε-perturbed game by requiring each player to imagine that he alone has an ε impact on the societal choice. Then, the set of SE consists of limits of Nash equilibrium distributions of ε-perturbed games when ε tends to 0. It needs to be pointed out that in the ε-perturbed game, players are not rational as in Selten (1975). This is because each player thinks that he alone has an ε impact on the societal choice, and does not contemplate that others do the same consideration.

After proving the existence of SE distributions under standard assumptions (e.g., Mas-Colell, 1984) we show that the SE is a refinement of Nash equilibrium. Moreover, using the representation results of Khan and Sun (1995), Carmona (2008) and Carmona and Podczeck (2009), it is established that this analysis can be extended to strategy profiles whenever either one of the following holds: (1) the action space of every player is countable; or (2) the set of possible types of players is countable; or (3) the space of players is super-atomless.

The impact of focusing on SE is well illustrated in the above example: In the game where players choose either 0 or 1, there is only one SE which consists of almost all players choosing 1. Hence, the distribution of actions induced by the SE coincides with the distribution induced by the unique Nash equilibrium of the same game when played by a finite number of players.

A similar strong conclusion holds in the Nash's mass action game as well: A (finite) normal-form game is interpreted to consist of a finite number of positions (or islands), each characterized by a finite action space and a payoff function on the joint action space. One, then, imagines that the actual players in this game reside on one of those islands, players on the same island have identical payoffs and are equally likely to be chosen to play the game. Therefore, starting from the case where there is only one player on each island, we formulate associated replicas by symmetrically multiplying players on each island and assuming that each player on an island is equally likely to be selected. Hence, for any $k \in \mathbb{N}$, the k-replica game is one in which there are k players on each island who are equally likely to be selected to play the original game, and the payoff function and the action set of every player on an island are identical. It is, then, not difficult to see that for any $k \in \mathbb{N}$, a strategy is an equilibrium of the k-replica game if and only if the vector consisting of the average choices across players of a given island is a mixed strategy Nash equilibrium of the original game. However, this equivalence fails to hold in the limit case of a continuum of players on each island, each of whom are selected according to the Lebesgue measure. Indeed, in this case, no player can affect the average choice of the island they reside on, and thus, every strategy is a Nash equilibrium. However, when SE is employed, this equivalence is restored: We prove that a strategy profile in the non-atomic version is a SE if and only if the vector of the average choices across players on the same island is a mixed strategy Nash equilibrium of the original normal-form game.

Similar conclusions are reached in dynamic situations as well. After presenting the notion of strategic subgame perfect equilibrium (henceforth SSPE), we demonstrate that its use in the optimal taxation game of Levine and Pesendorfer (1995), instead of subgame perfect equilibrium (abbreviated by SPE), makes sure that the first-best can be obtained even with non-atomic players. Indeed, using the concept of SPE in non-atomic optimal taxation games, e.g. Chari and Kehoe (1989), the government cannot detect (thus, punish) individual deviations because one single agent cannot affect the societal choice, a phenomenon labeled as the "disappearance of information" by Levine and Pesendorfer (1995). Even though, the first-best is uniquely obtained in SPE in finite player versions of the same (extensive-form) game, it is well known that the second-best, the Ramsey Equilibrium, is the best possible with the use of SPE in non-atomic formulations. This, in turn, gives rise to discussions about whether or not the government may commit in order to achieve this particular payoff. Besides delivering a sharper conclusion that is not in "paradoxical" terms with that from finite player cases, this game is also of interest as it involves the use of SE with sequential rationality.

However, the set of SE does not equal the set of limit equilibria in general. In fact, we provide an example of a regular equilibrium of a smooth non-atomic game, hence of a SE, which fails to be a limit equilibrium. On the other hand, in the above examples, the notion of SE meets our desiderata of always existing and reproducing the (limit) properties of equilibria of the same game played by a large finite number of players.

It should be emphasized that our analysis is related to, but differs from that of Green (1980), Sabourian (1990), Levine and Pesendorfer (1995), and Carmona and Podczeck (2011) who try to justify the set of Nash equilibria of non-atomic games as limits of equilibria of large finite games with either noisy observations about deviating players or employing the ε-equilibrium concept. That is, we are not asking "when agents are negligible in large finite games", but rather analyzing equilibria of non-atomic games that are limits of equilibria of games where each player thinks that he alone is not negligible.

Section 2 describes the general framework of non-atomic games. In Section 3 we define the concept of SE and prove that it exists and is a refinement of Nash equilibrium. Section 4 considers regular equilibria of smooth non-atomic games. Finally, Section 5 involves Nash's mass action interpretation while Section 6 formalizes the notion of SSPE and displays its use in the optimal taxation game of Levine and Pesendorfer (1995).

2. Games with a measure space of players

In this section, we formally describe a class of games with a measure space of players. This class of games is a particular case of the model in Carmona and Podczeck (2014) although we follow Mas-Colell's (1984) distributional approach.

The set of players consists of a finite set $\bar{T}$ and a probability space $(\hat{T}, \hat{\Sigma}, \hat{\nu})$ such that $\{t\} \in \hat{\Sigma}$ for all $t \in \hat{T}$ and $\bar{T} \cap \hat{T} = \emptyset$. The set of atomic players is $\bar{T}$ and the set of atomless players is $\hat{T}$. Let $T = \bar{T} \cup \hat{T}$.

The action set of each player $t \in \bar{T}$ is denoted by X_t and we let $X = \prod_{t \in \bar{T}} X_t$ and also $X_{-t} = \prod_{t' \in \bar{T} \setminus \{t\}} X_{t'}$. We assume that X_t is a nonempty, compact and convex subset of a locally convex topological vector space for each $t \in \bar{T}$.

In order to accommodate general examples, such as the Nash's mass action game, we allow players' payoff functions to depend on the distribution of choices made by a finite number of subgroups of