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Risk analysis for a stochastic cash management model with two types of customers

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Abstract

A stochastic cash management system is studied in which the cash flow is modeled by the superposition of a Brownian motion with drift and a compound Poisson process with positive and negative jumps for “big” deposits and withdrawals, respectively. We derive explicit formulas for the distributions of the bankruptcy time, the time until bankruptcy or the reaching of a prespecified level, the maximum cash amount in the system, and for the expected discounted revenue generated by the system.

Keywords: Cash management; Brownian motion; Compound Poisson process; First-exit time; Bankruptcy; Revenue functional; Maximum cash amount.

1. Introduction

We study a stochastic model of a cash management system. Two types of customers have to be served: many “little” customers frequently deposit and withdraw small amounts of money; the cash fluctuations generated by them will be modeled by a Brownian motion. The transactions of the few “big” customers cause upward or downward jumps in the amount of cash held by the system. The arrival times of the big customers are assumed to form a Poisson process. The cash flow in and out of the system is not directly controllable; it can only be influenced by the specification of the parameters of the underlying stochastic processes.

We suppose that the total drift of the cash flow is negative so that level zero will eventually be reached or downcrossed. The cash fund process $X = (X(t)|t \geq 0)$ is assumed to start at some initial capital $X(0) = x > 0$, to stop at the bankruptcy time $T = \inf\{t \geq 0|X(t) \leq 0\}$, and in between to fluctuate as a Lévy process formed by a Brownian and a compound Poisson component.

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The main objective of this paper is to derive several characteristic functionals of the cash fund process described above. We will determine the distribution of the bankruptcy time T by means of its Laplace transform (LT) $E(\mathrm{e}^{-\beta T}|X(0)=x)$. A suitable functional to measure the revenue generated by the cash fund is

$$E\left(\int_0^T \mathrm{e}^{-\beta t} X(t)\,\mathrm{d}t|X(0)=x\right), \quad \beta \geq 0,$$

since β can be interpreted as a discount factor. We also obtain as a by-product the functional

$$E\left(\int_0^T \exp\{-\beta X(t)\}\,\mathrm{d}t|X(0)=x\right)/E(T|X(0)=x),$$

which can also be interpreted as "quasi-stationary" LT. This makes it possible to compute the quasi-stationary nth moment functional. Another variable of interest is the last record value $K=\sup\{X(t)|0\leq t\leq T\}$, the maximum amount of cash in the system. To compute the distribution of K, we need the distributions of the two-sided stopping times $\tau_y=\inf\{t\geq 0|X(t)\notin(0,x+y)\}$, $y>0$, and of $\sup_{t\geq 0}X(t)$, which are of independent interest. We will also determine the joint distribution of K and $X(T)$. The explicit expressions that we obtain for all these quantities may lay the ground for optimization purposes.

The cash flow model described above is similar to other stochastic storage systems such as queues, inventories, dams, and insurance risk models. The crucial novel feature is the presence of random jumps upwards and downwards. In the classical models random changes occur only in one direction, while the flow in the other direction is deterministic. Moreover, our approach adds a second new dimension by considering small and large inflows and outflows simultaneously. The cash flow process studied here can also be interpreted as the workload for a $GI/G/1$ queue in heavy traffic with occasional "big" customers causing random upward jumps and work removals by "negative customers" (Boucherie and Boxma, 1996; Gelenbe, 1991; Jain and Sigman, 1996). What is here called "bankruptcy", can, after some reformulation, be considered as a clearing action in inventory systems (Kim and Seila, 1993; Perry and Stadje, 1998; Serfozo and Stidham, 1978; Stidham, 1977, 1986).

Several papers relate certain storage systems to cash flow management. Harrison and Taksar (1983) consider impulse control policies: when the cash fund gets too large, the controller may choose to convert some of his cash into securities. When the amount of cash decreases below some limit, securities may be reconverted into cash. Harrison et al. (1983) model the cash fund as a Brownian motion reflected at the origin. Both papers focus on the proofs that their special policies are optimal regarding transaction costs. Perry (1997) extended the model of Harrison et al. (1983) by considering drift control for a two-sided reflected Brownian motion, also taking into account holding cost and unsatisfied demand cost functionals. The combination of a Brownian and a compound Poisson component as suggested here can make these models more realistic. Milne and Robertson (1996) study the behavior of a firm whose cash flow is determined by a diffusion process and which faces liquidation if the internal cash balance falls below some threshold value. They look for an optimal trade-off between the desire to pay dividends and the need to retain cash as a barrier against possible liquidation. Browne (1995) considers a firm with an uncontrollable cash flow and the possibility to invest in a risky stock. In Asmussen and Taksar (1997) and Asmussen and Perry (1998) jump diffusion models motivated by finance and general storage applications are studied. Further related papers in which the buffer content can be interpreted as a cash fund are Asmussen and Kella (1996), Radner (1998), Zipkin (1992a,b).

2. Analysis of the model

We now introduce our model in a formal manner. The cash amount in the system at time t is given by

$$X(t)=x+B(t)+Y(t), \quad t\geq 0,$$