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An auction-based market equilibrium algorithm for a production model

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Abstract

We present an auction-based algorithm for computing market equilibrium prices in a production model, in which producers have a single linear production constraint, and consumers have linear utility functions. We provide algorithms for both the Fisher and Arrow–Debreu versions of the problem.

Keywords: Market equilibrium; Auction; Algorithm; Production

1. Introduction

Two basic models of market equilibria, consisting of buyers and goods, were defined by Fisher [2] and Walras [20] in the 19th century. The celebrated paper of Arrow and Debreu [1] introduced a general production model and gave proofs of existence of equilibria in all these models. In the Arrow–Debreu model, each production vector (schedule) lies in a specified convex set; negative coordinates represent raw materials and positive coordinates represent finished goods. For the problem of computing the equilibrium prices, the classic work of Eisenberg and Gale [9] gave a convex program for computing equilibrium prices for Fisher's model for the case of linear utility functions. For the production model, convex programs were obtained [16,19,17,18] when the model is restricted to positive production vectors only. An assumption of a set of raw goods outside the current market justifies this restricted model.

Over the last few years, there has been a surge of activity within the theoretical computer science community on the question of computing equilibria [5,6,13,7,8,3,15,10,11,4,14]. Perhaps the most novel aspect of this work lies in the development of combinatorial algorithms for these problems. Two techniques have mainly been successful: the primal–dual schema [6] and auctions [10]. Algorithms developed using these two approaches iteratively raise prices until equilibrium prices are found. They have been successful for models satisfying weak gross substitutability, i.e., raising the price of one good does not decrease the demand of another good.

Here we give an auction-based algorithm for the following case of the production model: buyers have linear utilities for goods and producers have a single linear capacity constraint on their production schedule. As in the case of convex

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programs for the production model mentioned above, we assume that raw materials are external to the current market. For any given prices of goods, each producer chooses a feasible schedule that maximizes his profit and each buyer chooses a bundle of goods that maximizes her utility. The problem is to find prices such that the market clears, i.e., all goods produced are bought and there is no deficiency of goods.

From the point of view of designing an algorithm that iteratively adjusts prices, an important difference between the consumer models of Fisher and Walras and the production models is that in the former the amount of each good available is fixed, while in the latter this changes with the prices of goods. So, while the algorithm is adjusting prices to dispose of existing goods, the amounts of goods available are also changing. However, it is easy to see that our production model (with a single constraint) satisfies weak gross substitutability in the following sense: raising the price of a good does not decrease the demand nor increase the production of another good. As a consequence, an algorithm that only increases prices can in principle arrive at the equilibrium. Indeed, our auction-based algorithm never needs to decrease prices. The actions taken by the auction-based algorithm are natural and correspond to actions performed in real markets. This is the first auction-based algorithm for such a setting.

2. The production model

Consider a market with q Producers and b Consumers, and m goods. Producer s produces $z_{sj} \geq 0$ amount of good j and his production schedule is constrained by a linear inequality of the form (5) below, where the $a_{sj} \geq 0$ and $K_s \geq 0$. The producers aim to sell the goods they produce, and they have utility for money. Consumer i has an initial endowment of e_i units of money, buys $x_{ij} \geq 0$ amount of good j, and has a linear utility function for the goods (1).

When the prices of the goods are fixed at (p_j), consumers and producers will buy and produce so as to satisfy the following programs, termed CLP and PLP:

$$\text{Maximize: } \sum_{1 \leq j \leq m} v_{ij} x_{ij} \tag{1}$$

$$\text{Subject to: } \sum_{1 \leq j \leq m} x_{ij} p_j \leq e_i \tag{2}$$

$$\forall j : \; x_{ij} \geq 0 \tag{3}$$

$$\text{Maximize: } \sum_{1 \leq j \leq m} p_j z_{sj} \tag{4}$$

$$\text{Subject to: } \sum_{1 \leq j \leq m} z_{sj} a_{sj} \leq K_s \tag{5}$$

$$\forall j : \; z_{sj} \geq 0. \tag{6}$$

Defining α_i as the bang-per-buck for consumer i, and γ_s as the profit rate for producer s, and using duality theory we can write consumer and producer conditions for optimality as follows:

$$\forall i : \alpha_i > 0 \Rightarrow \sum_j x_{ij} p_j = e_i \tag{7}$$

$$\forall j : \alpha_i p_j \geq v_{ij} \tag{8}$$

$$\forall j : x_{ij} > 0 \Rightarrow \alpha_i p_j = v_{ij} \tag{9}$$

$$\forall i : \alpha_i \geq 0 \tag{10}$$

$$\forall s : \gamma_s > 0 \Rightarrow \sum_j z_{sj} a_{sj} = K_s \tag{11}$$

$$\forall j : \gamma_s \geq p_j / a_{sj} \tag{12}$$

$$\forall s, j : z_{sj} > 0 \Rightarrow \gamma_s = p_j / a_{sj} \tag{13}$$

$$\forall s : \gamma_s \geq 0. \tag{14}$$

An equilibrium is a price vector (p_j) s.t. there are productions z_{sj} and allocations x_{ij} s.t. conditions ((7)–(14)) hold and furthermore all produced goods are sold:

$$\forall j \; : \sum_i x_{ij} = \sum_s z_{sj}. \tag{15}$$