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Computing violated sets in a market equilibrium problem with constant prices

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Abstract Consider Fisher's and Arrow-Debreu's market equilibrium models for a linear utilities case consisting of a set B of buyers and a set G of divisible goods. Supposing that a vector of prices $P = (p_1, \ldots, p_{|G|})$ for goods is given and there are some buyers with surplus money, but, by the politics of the market, prices P are constant and cannot be changed in order to compute an equilibrium. In this paper, a set of buyers with surplus money is called a violated set. First, we define a kind of violated set called maximum mean, then compute a set of buyers with maximum surplus money corresponding to the definition of a maximum mean violated set. We show a maximum mean set is found in $O(mn \log(n^2/m))$ time, where $n = |B| + |G|$, and m is the number of pairs (i, j), such that buyer i has some utility for purchasing goods j.

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1. Introduction

A market consisting of a set B of buyers and a set G of divisible goods is given. We are given for each buyer i the amount e_i of money she possesses, and for each good, j, the amount b_j of goods. Let u_{ij} denote the utility derived by i on obtaining a unit amount of goods j. Let $P = (p_1, \ldots, p_{|G|})$ denote a vector of prices. If, at these prices buyer, i is given goods j, she derives, u_{ij}/p_i amount of utility per unit amount of money spent. Define $\alpha_i = \max_j \; u_{ij}/p_j$. Clearly, buyer i will be happiest with goods that maximize u_{ij}/p_j. This motivates defining a bipartite graph $D = (G, B)$, which, for each $i \in B$ and $j \in G$, (i, j) is an edge in D iff $\alpha_i = u_{ij}/p_j$.

Computing the largest amount of goods that can be sold, without exceeding the budgets of buyers or the amount of goods available (assumed unit for each item of goods), can be accomplished by computing max-flow in the following network: Direct the edge of D from G to B and assign a capacity of infinity to all these edges. Introduce source vertex s, sink vertex t, a directed edge from s to each vertex $j \in G$ with a capacity of p_j, and a directed edge from each vertex $i \in B$ to t with a capacity of e_i. This network is clearly a function of the current prices, P, and defined by $N(P)$. An *equilibrium* is obtained w.r.t. the prices P iff $(s, G \cup B \cup t)$ and $(s \cup G \cup B, t)$ are two min-cuts in $N(P)$. Fisher's and Arrow–Debreu's market equilibrium models are the two fundamental models within mathematical economics. In both models, the purpose is to compute an equilibrium.

In Fisher' model [1], all initial endowments are in dollars: each buyer, i, has a fixed amount of money, e_i, and it does not change by increasing or decreasing the prices. Devanur et al. [2] gave the first polynomial time algorithm for computing an equilibrium, using $O(n^4(\log n + n \log U_{\max} + \log M))$ max-flow computations, where M depends on the endowments and $U_{\max}$ is the maximum utility. Recently, Orlin [3] developed the first strongly polynomial time algorithm for finding the market equilibrium. It runs in $O(n^4 \log n)$ time.

Arrow–Debreu's model [4] considers a more general model in which each buyer, i, starts with an initial endowment $\{e_{i1}, e_{i2}, \ldots, e_{i|G|}\}$ of goods, where e_{ij} is the initial proportion of goods j possessed by buyer i. If P is a vector of prices for the goods, then the value of the goods for buyer i is $e_i(P) = \sum_{j \in G} e_{ij} p_j$. Jain [5] gave a polynomial time algorithm for computing an equilibrium for this model using the ellipsoid algorithm. Ye [6] developed a faster polynomial time algorithm for computing market equilibrium using interior point algorithms. The algorithm in [6] runs in $O(n^4 L)$ time, where L is the bit-length of the input data U_{ij} (where U_{ij} is the utility of buyer i purchasing all of goods j). It still remains an open problem on

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how to compute the Arrow–Debreu market equilibrium exactly using combinatorial techniques (other than the ellipsoid algorithm). However, the Arrow–Debreu market equilibrium can be approximated using combinatorial methods. Jain et al. [7] gave the first Fully Polynomial Time Approximation Scheme (FPTAS) for Arrow–Debreu's model with linear utilities. They gave a combinatorial method to compute an ϵ-approximate solution, which runs in $O(1/\epsilon)$ calls of the algorithm in [2]. Devanur and Vazirani [8] improved the running time to $O((n^7/\epsilon)\log n/\epsilon)$. This running time avoids dependence on the size of the integers in the problem instance. Garg and Kapoor [9] relaxed the definition of approximation by permitting purchases to violate their optimality conditions by ϵ. Under this revised notion of approximation, they developed an $O((n^3/\epsilon)\log n/\epsilon)$ time algorithm. More recently, Ghiyasvand and Orlin [10] developed an approximation algorithm that runs in $O(n^3/\epsilon)$ time, using a new definition of approximation.

Consider the following Invariants:

Invariant-1. The cut $(s, G \cup B \cup t)$ is a min-cut in $N(P)$.
Invariant-2. The cut $(s \cup G \cup B, t)$ is a min-cut in $N(P)$.

Therefore, an equilibrium is obtained w.r.t. prices P iff the both Invariant-1 and Invariant-2 are satisfied. Supposing that Invariant-1 is satisfied, but Invariant-2 is not. Thus, there are some buyers with surplus money w.r.t. the current prices, P. For satisfying Invariant-2, we should increase the prices. In this paper, we suppose the current prices P cannot be changed. This case can occure when, by the politics of a market, the current prices should be constant. We call a set of buyers with surplus money as a violated set and are looking for the violated sets with maximum surplus money. Knowing these buyers can be helpful so that they are sent to another market or are not entered to the current market.

In this paper, we define a set of such buyers as a maximum mean set. For finding a maximum mean set, we define a flow in $N(P)$ called a min–max surplus flow and show the relationship between a min–max surplus flow and a maximum mean set, which says a maximum mean set can be computed using a min–max surplus flow. We first prove a min–max surplus flow can be computed using, at most, $|G|$ maximum flow computations, then we show, using parametric networks, that it is computed in $O(mn\log(n^2/m))$ time, where $n = |B| + |G|$, and m is the number of pairs (i, j), such that buyer i has some utility for purchasing goods j.

This paper consists of two sections in addition to the introduction. Section 2 defines violated sets and maximum mean sets. A method to compute a maximum mean set is presented in Section 3.

2. Violated sets

In this section, we first present a necessary and sufficient condition to satisfy Invariant-2 and define a new problem using this condition. For each $T \subseteq B$, define its money $m(T) = \sum_{j\in T} e_j$. W.r.t. prices P, for $S \subseteq G$, define $m(S) = \sum_{i\in S} p_i$. For $T \subseteq B$ and $S \subseteq G$, define its neighborhood in $N(P)$ by:

$$\Gamma^{-1}(T) = \{i \in G \mid \exists j \in T \text{ s.t. } (i, j) \in N(P)\},$$
$$\Gamma(S) = \{j \in B \mid \exists i \in S \text{ s.t. } (i, j) \in N(P)\}.$$

Each edge from G to B has a capacity of infinity, so, for each $T \subseteq B$, $m(\Gamma^{-1}(T))$ is the maximum value of flow entering into T.

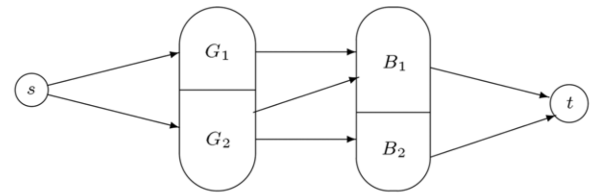

Figure 1: The minimum cut $(s \cup G_1 \cup B_1, t \cup G_2 \cup B_2)$.

Lemma 1. *For given prices P, $(s \cup G \cup B, t)$ is a minimum cut in the network $N(P)$ iff:*

$$\forall T \subseteq B : m(\Gamma^{-1}(T)) \geq m(T).$$

Proof. It is obvious that if $(s \cup G \cup B, t)$ is a minimum cut, then, for each $T \subseteq B$, the maximum flow entering into T should be equal or bigger than $m(T)$, i.e. $m(\Gamma^{-1}(T)) \geq m(T)$.

Now, supposing that $\forall T \subseteq B : m(\Gamma^{-1}(T)) \geq m(T)$, we show $(s \cup G \cup B, t)$ is a minimum cut in $N(P)$. Assume $(s \cup G_1 \cup B_1, t \cup G_2 \cup B_2)$ is a min-cut in $N(P)$, with $G_1, G_2 \subseteq A$ and $B_1, B_2 \subseteq B$. Because each edge from G to B has a capacity of infinity and the value of each maximum flow from s to t is not infinity, we get $\Gamma^{-1}(B_2) = G_2$ (see Figure 1). Thus, the capacity of the cut $(s\cup G_1\cup B_1, t\cup G_2\cup B_2)$ is $m(G_2)+m(B_1)$. It is obvious that the capacity of the cut $(s\cup G\cup B, t)$ is $m(B_1)+m(B_2)$. On the other hand, by assumption, we have $m(B_2) \leq m(\Gamma^{-1}(B_2)) = m(G_2)$, which means $m(B_1) + m(B_2) \leq m(G_2) + m(B_1)$. Hence, $(s \cup G \cup B, t)$ is a minimum cut. □

For given prices P and each set $T \subseteq B$, we define the *value of set T* by:

$$V^P(T) = m(T) - m(\Gamma^{-1}(T)).$$

If Invariant-1 is satisfied, then, by Lemma 1, an equilibrium is obtained w.r.t. prices P, iff, for every set, $T \subseteq B : V^P(T) \leq 0$. We define a set $T \in B$ as a *violated set* if $V^P(T) > 0$. If Invariant-1 is satisfied but an equilibrium is not obtained w.r.t. prices P, then Lemma 1 says that there are some violated sets in $N(P)$ w.r.t. the current prices P. We call *the mean value of set T* by:

$$\overline{V}^P(T) = \frac{V^P(T)}{|T|} = \frac{m(T) - m(\Gamma^{-1}(T))}{|T|},$$

and *a maximum mean set* $\overline{T}^*$ is defined by:

$$\overline{V}^P(\overline{T}^*) = \max_{T\in B} \overline{V}^P(T).$$

We define $\overline{V}(P) = \overline{V}^P(\overline{T}^*)$ as *the value of a maximum mean set* w.r.t. prices P. Using these definitions and Lemma 1, we get the following conclusion.

Conclusion 1. If the Invariant-1 is satisfied, an equilibrium is obtained w.r.t. prices P iff:

(a) For every set $T \subseteq B$: $\overline{V}^P(T) \leq 0$, or
(b) For every set $T \subseteq B$: $V^P(T) \leq 0$.

Computing a maximum mean set helps to know the set of buyers with maximum surplus money w.r.t. the current prices, P. This information is helpful when there is another market, but a set of buyers with maximum surplus money should be sent there.

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