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Stability of the Kalman Filter for Output Error Systems ⋆

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Abstract: Optimality and numerical efficiency are well known properties of the Kalman filter, whereas its stability property, though equally classical and important in practice, is less often mentioned in the recent literature. The stability of the Kalman filter is usually ensured by the uniform complete controllability *regarding the process noise* and the uniform complete observability of linear time varying systems. Such classical results cannot be applied to *output error systems*, in which the process noise is totally absent. It is shown in this paper that the uniform complete observability is sufficient to ensure the stability of the Kalman filter applied to time varying output error systems, regardless of the stability of the considered system itself.

Keywords: Stability, Kalman filter, Output Error System.

1. INTRODUCTION

The well known Kalman filter has been extensively studied and is being applied in many different fields (Anderson and Moore (1979); Jazwinski (1970); Zarchan and Musof (2005); Grewal and Andrews (2008)). The purpose of the present paper is to study the stability of the Kalman filter in a particular case rarely covered in the literature: the absence of process noise in the state equation of a linear time varying (LTV) system. Such systems are known as *output error* (OE) systems. Though typically process noise and output noise are both considered in Kalman filter applications, the case with *no* process noise is of particular interest when state equations come from physical laws that are believed accurate enough. It is also important for the application of the Kalman filter to OE system identification (Goodman and Dudley (1987); Forssell and Ljung (2000)).

While the optimal property of the Kalman filter is frequently recalled, its stability property is less often mentioned in the recent literature. The classical stability analysis is based both on the uniform complete controllability *regarding the process noise* and on the uniform complete observability of the considered system (Kalman (1963); Jazwinski (1970)). In the case of OE systems, there is no process noise at all in the state equation, hence the controllability regarding the process noise cannot be fulfilled, and the classical stability results are not applicable. The present paper aims at completing this missing case.

The optimal state estimation realized by the Kalman filter is usually viewed as a trade-off between the uncertainties in the state equation and in the output equation. In an OE system, the state equation is assumed noise-free. This point of view suggests that the state estimation should solely rely on the state equation, provided that the initial state of the OE system is *exactly* known. In practice the Kalman filter remains useful when the initial state is not exactly known or when the OE system is unstable. Of course, if the state of an unstable system diverges, so does its state estimate by the Kalman filter. Typically in practice, unstable systems are stabilized by feedback controllers so that the system state remains bounded. The Kalman filter can be applied either to the controlled system itself or to the entire closed loop system. In the latter case, the controller must be linear and completely known, excluding the saturation protection and any other nonlinearities.

The classical *optimality* results of the Kalman filter are also valid in the case of OE systems (Jazwinski, 1970, chapter 7). Nevertheless, it remains to complete the stability analysis, as the classical results are not applicable here.

The main results presented in this paper are as follows. Under the uniform complete observability condition, the dynamics of the Kalman filter applied to a LTV OE system is asymptotically stable, *regardless of the stability of the system itself.* The boundedness of the solution of the Riccati equation, which ensures the boundedness of the Kalman gain, is also proved under the same condition. These results are quite similar to the classical results (Kalman (1963); Jazwinski (1970)), which exclude the case of OE systems.

For linear time invariant (LTI) systems, it is a common practice to design the Kalman filter by solving an algebraic Riccati equation (in contrast to dynamic differential Riccati equation for general LTV systems as considered in the present paper). In this case, the controllability and observability conditions can be replaced by the weaker stabilizability and detectability conditions (Laub (1979); Arnold and Laub (1984)). Some preliminary results about

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LTI OE systems have been presented in (Ni and Zhang (2013)).

The rest of the paper is organized as follows. Some preliminary elements are introduced in Section 2. The problem considered in this paper is formulated in Section 3. The properties of the solution of the Riccati equation are analyzed in Section 4. The stability of the Kalman filter for OE systems is established in Section 5. Some numerical examples are presented in Section 6. Finally, concluding remarks are drawn in Section 7.

2. DEFINITIONS

Let us shortly recall some definitions about LTV systems, which are necessary for the following sections.

Let m and n be any two positive integers. For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm. For a matrix $A \in \mathbb{R}^{m\times n}$, $\|A\|$ denotes the matrix norm induced by the Euclidean vector norm, which is equal to the largest singular value of A and known as the spectral norm when $m = n$. Then $\|Ax\| \leq \|A\|\|x\|$ for all $A \in \mathbb{R}^{m\times n}$ and all $x \in \mathbb{R}^n$. For two real square symmetric positive definite matrices A and B, $A > B$ means $A-B$ is positive definite.

Let $A(t) \in \mathbb{R}^{m\times n}$ be defined for $t \in \mathbb{R}$. It is said (upper) bounded if $\|A(t)\|$ is bounded.

Consider the homogeneous LTV system

$$\frac{dx(t)}{dt} = A(t)x(t) \tag{1}$$

with $x(t) \in \mathbb{R}^n$ and $A(t) \in \mathbb{R}^{n\times n}$, and let $\Phi(t, t_0)$ be the associated state transition matrix such that, for all $t, t_0 \in \mathbb{R}$, $d\Phi(t, t_0)/dt = A(t)\Phi(t, t_0)$ and $\Phi(t, t) = I_n$ with I_n denoting the $n \times n$ identity matrix.

Definition 1. System (1) is *Lyapunov stable* if there exists a positive constant γ such that, for all $t, t_0 \in \mathbb{R}$ satisfying $t \geq t_0$, the following inequality holds

$$\|\Phi(t, t_0)\| \leq \gamma. \tag{2}$$

□

This definition concerns the boundedness of the state vector, whereas the following definition ensures its convergence to zero.

Definition 2. System (1) is *asymptotically stable* if it is Lyapunov stable and if the following limiting behavior holds

$$\lim_{t\to+\infty} \|x(t)\| = 0. \tag{3}$$

□

The last concept to be recalled here is about the observability of LTV systems, following (Kalman (1963)).

Definition 3. The matrix pair $[A(t), C(t)]$ with $A(t) \in \mathbb{R}^{n\times n}$ and $C(t) \in \mathbb{R}^{m\times n}$ is *uniformly completely observable* if there exist positive constants τ, ρ_1 and ρ_2 such that, for all $t \in \mathbb{R}$, the following inequalities hold

$$\rho_1 I_n \leq \int_{t-\tau}^{t} \Phi^T(s, t)C^T(s)R^{-1}(s)C(s)\Phi(s, t)ds \tag{4}$$

$$\leq \rho_2 I_n \tag{5}$$

with some bounded symmetric positive definite matrix $R(s) \in \mathbb{R}^{m\times m}$ (typically the covariance matrix of the output noise in a stochastic state space system). □

3. PROBLEM FORMULATION AND ASSUMPTIONS

In this section is first recalled the classical Kalman filter under its usual assumptions, before the presentation of the particular case of OE systems considered in this paper.

3.1 Kalman filter in the usual case

The Kalman filter in continuous-time is usually applied to LTV systems modeled by

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + Q^{\frac{1}{2}}(t)d\omega(t) \tag{6a}$$

$$dy(t) = C(t)x(t)dt + R^{\frac{1}{2}}(t)d\eta(t) \tag{6b}$$

where $t \in \mathbb{R}$ represents the time, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^l$ the bounded input, $y(t) \in \mathbb{R}^m$ the output, $\omega(t) \in \mathbb{R}^n, \eta(t) \in \mathbb{R}^m$ are two independent Brownian processes with identity covariance matrices, $A(t), B(t), C(t), Q(t), R(t)$ are real matrices of appropriate sizes. The matrix $Q(t)$ is symmetric positive semi-definite, and $R(t)$ is symmetric positive definite. The notations $Q^{\frac{1}{2}}(t)$ and $R^{\frac{1}{2}}(t)$ denote respectively the symmetric positive (semi)-definite matrix square roots of $Q(t)$ and $R(t)$. The initial state $x(t_0) \in \mathbb{R}^n$ is a random vector following the Gaussian distribution $x(t_0) \sim \mathcal{N}(x_0, P_0)$ with $x_0 \in \mathbb{R}^n$ and $P_0 \in \mathbb{R}^{n\times n}$.

The Kalman filter for this LTV system writes

$$d\hat{x}(t) = A(t)\hat{x}(t)dt + B(t)u(t)dt + K(t)(dy(t) - C(t)\hat{x}(t)dt) \tag{7a}$$

$$K(t) = P(t)C^T(t)R^{-1}(t) \tag{7b}$$

$$\frac{d}{dt}P(t) = A(t)P(t) + P(t)A^T(t) - P(t)C(t)^T R^{-1}(t)C(t)P(t) + Q(t) \tag{7c}$$

$$\hat{x}(t_0) = x_0, \quad P(t_0) = P_0 \tag{7d}$$

where the solution of the Riccati equation (7c) is a matrix function $P(t) \in \mathbb{R}^{n\times n}$ and the Kalman gain $K(t) \in \mathbb{R}^{n\times m}$.

The optimal properties of the Kalman filter are well known. It is less well known, though equally classical and important, that the solution $P(t)$ of the Riccati equation is bounded and that the dynamics of the Kalman filter is stable, provided the matrix pair $[A(t), Q^{\frac{1}{2}}(t)]$ is uniformly completely controllable and the matrix pair $[A(t), C(t)]$ is uniformly completely observable (Kalman (1963); Jazwinski (1970)). These are obviously crucial properties in practice for online applications.

It is worth mentioning that, in (Kalman (1963); Jazwinski (1970)), these classical results are based on an important lemma, which turns out to be incorrect, as pointed out independently by the authors of (Delyon (2001); Pengov et al. (2001)). Fortunately, the mistake has been repaired in these more recent references so that the main classical results remain correct.

3.2 Output error systems and Kalman filter

In the case of OE systems, the process noise is absent from the state equation, hence the general LTV system (6) becomes

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt \tag{8a}$$

$$dy(t) = C(t)x(t)dt + R^{\frac{1}{2}}(t)d\eta(t). \tag{8b}$$