Robust Static Output Feedback Controllers via Robust Stabilizability Functions

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***Abstract*—This technical note addresses the design of robust static output feedback controllers that minimize a polynomial cost and robustly stabilize a system with polynomial dependence on an uncertain vector constrained in a semialgebraic set. The admissible controllers are those in a given hyper-rectangle for which the system is well-posed. First, the class of robust stabilizability functions is introduced, i.e., the functions of the controller that are positive whenever the controller robustly stabilizes the system. Second, the approximation of a robust stabilizability function with a controller-dependent lower bound is proposed through a sums-of-squares (SOS) program exploiting a technique developed in the estimation of the domain of attraction. Third, the derivation of a robust stabilizing controller from the found controller-dependent lower bound is addressed through a second SOS program that provides an upper bound of the optimal cost. The proposed method is asymptotically non-conservative under mild assumptions.**

***Index Terms*—Robust control, robust stabilizability function, SOS polynomial, uncertain system.**

I. Introduction

A key problem in systems with uncertainty consists of designing robust stabilizing controllers, in particular feedback controllers that, without requiring to measure the uncertainty, ensure robust stability (i.e., stability for all admissible uncertainties) of the closed-loop system. Numerous approaches have been proposed for robust stability analysis of systems affected by parametric uncertainties, mainly based on the use of Lyapunov functions and convex optimization problems with linear matrix inequalities (LMIs), see, e.g., [2], [6], [8], [9], [11], [15]. Unfortunately, these approaches unavoidably lead to nonconvex optimization whenever applied to robust control design. In fact, whenever a controller to be designed is present, the LMIs generally become bilinear matrix inequalities (BMIs) in the unknown Lyapunov function and controller. In order to cope with this issue, several approaches have been proposed, for instance based on the introduction of generalized multipliers and slack variables. Although conservative, these approaches are quite flexible as they allow one to cope with several performance requirements such as minimization of the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms. See, e.g., [1], [7], [13]. Other approaches have been proposed without the use of Lyapunov functions, such as [6] which provides conditions for robust stability, and [10] which estimates robust stability regions.

This technical note addresses the design of robust static output feedback controllers that minimize a polynomial cost and robustly stabilize a system with polynomial dependence on an uncertain vector constrained in a semialgebraic set. The admissible controllers are those in a given hyper-rectangle for which the system is well-posed. First, the class of robust stabilizability functions is introduced, i.e., the functions of the controller that are positive whenever the controller robustly stabilizes the system. Second, the approximation of a robust stabilizability function with a controller-dependent lower bound is proposed through a SOS program exploiting a technique developed in the estimation of the domain of attraction. Third, the derivation of a robust stabilizing controller from the found controller-dependent lower bound is addressed through a second SOS program that provides an upper bound of the optimal cost. The proposed method is asymptotically non-conservative under mild assumptions. A conference version of this technical note (without the proofs and the convergence analysis) will appear as reported in [5].

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II. Problem Formulation

Notation: $\mathbb{R}$: real numbers; A': transpose; $\det(A)$: determinant; $adj(A)$: adjoint; $spec(A)$: set of eigenvalues; $col(A)$: column vector stacking the columns of A; $A>0$, $A\geq 0$: symmetric positive definite and symmetric positive semidefinite matrix; Hurwitz matrix: matrix with all eigenvalues having negative real part; $\deg(a)$: degree. Let us consider

$$\begin{cases} \dot{x}(t) = A(p)x(t) + B(p)u(t) \\ y(t) = C(p)x(t) + D(p)u(t) \end{cases} \tag{1}$$

where $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^{n_u}$, $y(t)\in\mathbb{R}^{n_y}$, $p\in\mathbb{R}^q$, and $A(p)$, $B(p)$, $C(p)$ and $D(p)$ are matrix polynomials. It is supposed that

$$p\in\mathcal{P} \tag{2}$$

$$\mathcal{P} = \{p\in\mathbb{R}^q : a_i(p)\geq 0, b_j(p)=0, i=1\ldots,n_a, \\ j=1,\ldots,n_b\} \tag{3}$$

where $a_i(p)$ and $b_j(p)$ are polynomials. This system is controlled via

$$u(t) = Ky(t) \tag{4}$$

where $K\in\mathbb{R}^{n_u\times n_y}$ is the controller to be determined. It is supposed that

$$K\in\mathcal{K} \tag{5}$$

where $\mathcal{K}\subset\mathbb{R}^{n_u\times n_y}$ is the hyper-rectangle

$$\mathcal{K} = \left\{K\in\mathbb{R}^{n_u\times n_y} : k_{ij}^-\leq k_{ij}\leq k_{ij}^+, \\ i=1,\ldots,n_u, j=1,\ldots,n_y\right\} \tag{6}$$

where $k_{ij}\in\mathbb{R}$ is the (i,j)-th entry of K, and k_{ij}^-, k_{ij}^+ are its lower and upper bounds. The closed-loop system (1)–(6) can be rewritten as

$$\begin{cases} \dot{x}(t) = A_{cl}(K,p)x(t) \\ K\in\mathcal{K}\cap\mathcal{K}_{wp} \\ p\in\mathcal{P} \end{cases} \tag{7}$$

$$A_{cl}(K,p) = A(p) + B(p)K\left(I - D(p)K\right)^{-1}C(p) \tag{8}$$

where $\mathcal{K}_{wp}$ is the set of controllers such that $A_{cl}(K,p)$ is well-posed. In particular, we say that $A_{cl}(K,p)$ is well-posed if

$$\left|\det\left(I - D(p)K\right)\right| \geq \rho_{wp} \quad \forall p\in\mathcal{P} \tag{9}$$

where $\rho_{wp}>0$ is an arbitrary small chosen threshold. Hence

$$\mathcal{K}_{wp} = \left\{K\in\mathbb{R}^{n_u\times n_y} : \text{(9) holds}\right\}. \tag{10}$$

The system (7) is said robustly stable if, for some $\rho_s\geq 0$

$$\Re(\lambda) < -\rho_s \quad \forall\lambda\in spec\left(A_{cl}(K,p)\right) \quad \forall p\in\mathcal{P}. \tag{11}$$

Problem: Establish the existence of a robust stabilizing controller for the system (7), i.e., the non-emptyness of

$$\mathcal{K}_s = \{K \in \mathcal{K} \cap \mathcal{K}_{wp} : \text{(11) holds}\}. \tag{12}$$

Also, we aim to determine a controller in $\mathcal{K}_s$ that minimizes a given polynomial cost $r(K)$, i.e.

$$r^* = \inf_{K \in \mathcal{K}_s} r(K). \tag{13}$$

Indeed, since $\mathcal{K}_s$ generally contains an infinite number of controllers whenever is non-empty, one might want to pick one according to some criterion. For instance, this can be the minimization of the Euclidean norm of the entries of K, since actuators with small gains can be preferable in real systems as they require less power.

III. Robust Stabilizability Functions

We say that $s : \mathbb{R}^{n_u \times n_y} \to \mathbb{R}$ is a robust stabilizability function over the set $\mathcal{K} \cap \mathcal{K}_{wp}$ for the system (7) if and only if

$$K \in \mathcal{K} \cap \mathcal{K}_{wp} \Rightarrow \begin{cases} s(K) > 0 & \text{if } K \in \mathcal{K}_s \\ s(K) \leq 0 & \text{otherwise.} \end{cases} \tag{14}$$

Here we show how to build a robust stabilizability function through the use of the Routh-Hurwitz criterion on the characteristic polynomial of the matrix $A_{cl}(K,p)$. Let us express $A_{cl}(K,p)$ as

$$A_{cl}(K,p) = \frac{\bar{A}_{cl}(K,p)}{\det(I - D(p)K)} \tag{15}$$

where $\bar{A}_{cl}(K,p)$ is the matrix polynomial

$$\begin{aligned}\bar{A}_{cl}(K,p) = \det(I - D(p)K)\, A(p) \\ + B(p)K adj\,(I - D(p)K)\, C(p)\end{aligned} \tag{16}$$

and $adj(I - D(p)K)$ is the adjoint matrix of $I - D(p)K$. In order to get rid of the denominator in (15), we need to consider two possible cases depending on its sign. To this end, let us define

$$\mathcal{T} = \begin{cases} \{0\} & \text{if } D(p) = 0 \\ \{0,1\} & \text{otherwise} \end{cases} \tag{17}$$

and the partition of $\mathcal{K}_{wp}$ given by

$$\mathcal{K}_{wp} = \bigcup_{\tau \in \mathcal{T}} \mathcal{U} \tag{18}$$

$$\begin{aligned}\mathcal{U} = \big\{K \in \mathbb{R}^{n_u \times n_y} : (-1)^\tau \det(I - D(p)K) \\ \geq \rho_{wp} \forall p \in \mathcal{P}\big\}.\end{aligned} \tag{19}$$

Let $z \in \mathbb{R}^{n_z}$ be the variable

$$z = \begin{pmatrix} col(K) \\ p \end{pmatrix}, \quad n_z = n_u n_y + q. \tag{20}$$

For $\tau \in \mathcal{T}$, let us define

$$W(z) = (-1)^\tau \left(\bar{A}_{cl}(K,p) + \rho_s \det(I - D(p)K)\, I\right) \tag{21}$$

and its characteristic polynomial

$$v(\lambda, z) = \det(\lambda I - W(z)) \tag{22}$$

where $\lambda \in \mathbb{C}$. Let us express $v(\lambda, z)$ as

$$v(\lambda, z) = \lambda^n + \sum_{i=0}^{n-1} d_i(z) \lambda^i \tag{23}$$

where $d_i(z)$ are polynomials. Let us write the Routh-Hurwitz table of $v(\lambda, z)$ as

$$\begin{aligned} &e_{00}(z) = 1, \quad e_{01}(z) = d_{n-2}(z), \ldots \\ &e_{10}(z) = d_{n-1}(z), \quad e_{11}(z) = d_{n-3}(z), \ldots \\ &e_{ij}(z) = \frac{e_{i-10}(z) e_{i-2j+1}(z) - e_{i-20}(z) e_{i-1j+1}(z)}{e_{i-10}(z)} \end{aligned} \tag{24}$$

for $i = 2, \ldots, n$, $j = 0, 1, \ldots$ We have that $e_{ij}(z)$ can be expressed as

$$e_{ij}(z) = \frac{\bar{e}_{ij}(z)}{\hat{e}_{ij}(z)}, \quad \hat{e}_{ij}(z) = \prod_{l = i-1, i-3, \ldots} \bar{e}_{l0}(z). \tag{25}$$

where $\bar{e}_{ij}(z)$ and $\hat{e}_{ij}(z)$ are polynomials. Let us define the set

$$\mathcal{N} = \{i = 0, \ldots, n : \bar{e}_{i0}(z) \text{ is a non-positive constant}\} \tag{26}$$

and let $f_m(z)$, $m = 1, \ldots, n_f$, be the non-constant polynomials among $\bar{e}_{i0}(z)$, $i = 0, \ldots, n$.

Theorem 1: Let $\tau \in \mathcal{T}$. If $\mathcal{N} \neq \emptyset$, then

$$\text{(11) does not hold for any } K \in \mathcal{U}. \tag{27}$$

Hence, suppose that $\mathcal{N} = \emptyset$, and let us define

$$s(K) = \inf_{\substack{p \in \mathcal{P} \\ m = 1, \ldots, n_f}} f_m(z). \tag{28}$$

Then

$$\begin{cases} s(K) > 0 \\ K \in \mathcal{K} \cap \mathcal{U} \end{cases} \Rightarrow K \in \mathcal{K}_s. \tag{29}$$

Moreover, if $\mathcal{P}$ is compact, this condition holds in both directions, i.e., $s(K)$ is a robust stabilizability function over the set $\mathcal{K} \cap \mathcal{U}$ for the system (7).

Proof: Let us consider $\mathcal{N} = \emptyset$, and suppose that $s(K) > 0$ for $K \in \mathcal{K} \cap \mathcal{U}$. This implies that

$$\bar{e}_{i0}(z) > 0 \quad \forall i = 0, \ldots, n \;\forall p \in \mathcal{P}.$$

From (25) one obtains

$$e_{i0}(z) > 0 \quad \forall i = 0, \ldots, n \;\forall p \in \mathcal{P}$$

hence implying that

$$W(z) \text{ is Hurwitz } \forall p \in \mathcal{P}.$$

Since $K \in \mathcal{U}$ one has

$$(-1)^\tau \det(I - D(p)K) \geq \rho_{wp} \quad \forall p \in \mathcal{P}$$

and, hence,

$$\frac{W(z)}{(-1)^\tau \det(I - D(p)K)} \text{ is Hurwitz } \forall p \in \mathcal{P}.$$