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Observer design for inherently nonlinear systems with lower triangular structure

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Abstract— **A new observation procedure is proposed for a wide class of single output observable nonlinear systems written in lower triangular form. First, we give the n-th order time-varying differentiator that robustly estimates, in asymptotic manner, the higher derivatives of any model-free continuously differentiable signal. This n-th order differentiator is a generalization of the time-varying differentiator proposed by the author in [1], [2] and [3]. By using an appropriate change of variables, it is shown that the boundedness of the signal to be differentiated is not necessary for the convergence of the differentiator. Based on the fact that systems written in triangular form are algebraically observable then, the system states can be reproduced through a static diffeomorphism that involves the system input, the system output, and their respective higher derivatives. It is shown that the global convergence of the n-th order differentiator implies the asymptotic convergence of the system states without imposing any restrictive condition on the form of nonlinearities.**

***Index Terms*— Nonlinear observer design; Adaptive estimation; Time-varying systems; Signal differentiation.**

I. INTRODUCTION

STATE estimation of highly nonlinear systems is a long-standing and challenging problem that has been addressed with different looks. The complexity of state reconstruction from the input and the output measurements depends on the system nonlinearities, the nature of the input that may render the system unobservable, and the form of the system output which plays a key role in the stability of the observation error. Until now, there is no unique straightforward method to design an observer for a given nonlinear system. However, under certain conditions, numerous solutions do exist for special forms of systems. By exploiting the structure of the system being observed, the boundedness of the system states or the Lipschitz property of the system nonlinearities, many strategies have been employed to build an observer. Error-linearization-based algorithms [4], [5], [6], [7], Lyapunov design procedures [8], and sliding-mode observer design [9], [10] are among the systematic procedures that have shown satisfactory performances. The reader can also find other challenging procedures as numerical methods [11], neural-network observation techniques [12], algebraic nonlinear observer design [1], and circle-criterion observation methods [13], [14]. When the system fails to be put in certain form of observability, high-gain observer design reveals as a powerful method that is often used to reconstruct the system states under the assumption that the vector nonlinearity is globally or locally Lipschitz, see [15], [16], [17], [18], [19]. However, the Lipschitz constraint is not always verified and prevents generally the global convergence of the high-gain observer. Moreover, the existence of the observer gain is conditioned by the value of the Lipschitz constant which is generally required to be small enough, see [19] for more details. Even though the circle-criterion observer design is conceptually free form the information of the Lipschitz constant [20], [14], this interesting design remains limited to systems with positive-gradient nonlinearities. In this note, a new observation method is given for state estimation of a general class of nonlinear systems satisfying the complete uniform observability condition. The main features of the proposed design are summarized in the following points.

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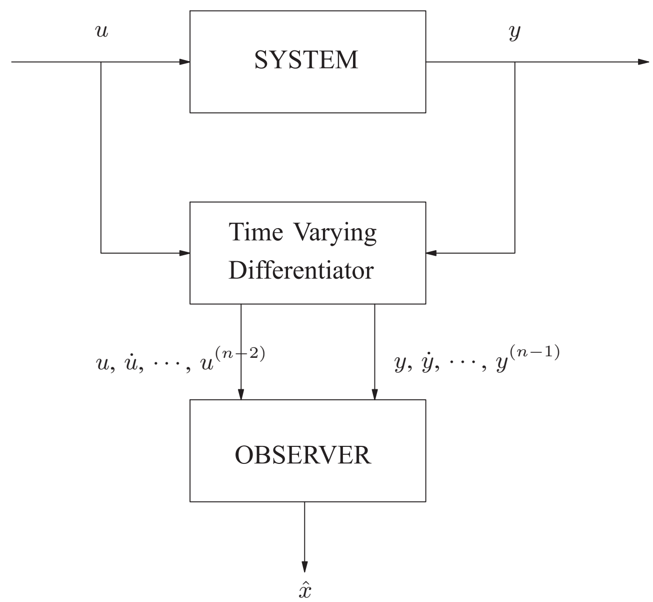

Fig. 1. The scheme of the nonlinear observer

- Robustly estimate the higher derivatives of any differentiable measured signal without incorporating its model or imposing the boundedness of the signal or its respective higher derivatives;
- The proposed n-th order differentiator is a generalization of constant-gain differentiators written in controllable canonical form discussed in [2];
- The design procedure is free from any restrictive condition as the Lipschitz or the Hölder conditions generally imposed in high-gain observer design;
- The nonlinearities are not subject to any restrictive condition whenever the uniform observability condition is satisfied;

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- The dynamics of the adaptive algebraic observer is not a copy of the original model with output correction term;
- The convergence of the observation error is global and exponential.

To conceive the dynamics of the whole observer, we start by writing the system states as static algebraic expressions of the input, the output, and their respective higher derivatives. Subsequently, all the variables of the static diffeomorphism, that relate the system unmeasured states to the higher derivatives of the input and the output, are asymptotically estimated. Illustrative example showing the main features of the novel design is discussed. Throughout this paper, we note by $\mathbb{R}$ the set of real numbers. The notation $A > 0$ (resp. $A < 0$) means that the matrix A is positive definite (resp. negative definite). I_n is the identity matrix of appropriate dimension and A' denotes the matrix transpose of A. We note by $\triangleq$ any equality by definition. $\dot{x}$ stands for the time-derivative of the vector x with respect to time and C_n^k stands for the binomial coefficient.

II. System description and observability analysis

Consider the class of dynamical systems written in the lower-triangular form:

$$
\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_1, u),\\
&\vdots\\
\dot{x}_i &= x_{i+1} + f_i(x_1, x_2, \cdots, x_i, u),\\
&\vdots\\
\dot{x}_n &= f_n(x_1, \cdots, x_n, u),\\
y &= x_1,
\end{aligned}
\tag{1}
$$

where $u \in \mathbb{R}^m$ is the system input and $y \in \mathbb{R}$ is the system output. In order to complete the description of the considered system, let us begin by considering the following assumptions.

Assumption 1: Each nonlinearity $f_i(x_1, x_2, \cdots, x_i, u)$, $1 \leq i \leq n$ is a well-defined nonlinearity with respect to their variables x_1, x_2, $\cdots$, x_i, u.

Assumption 2: For a given bounded input $u \in \mathbb{R}^m$, the system states do not leave any compact set. In other words, the system trajectories are well-defined for all $t \geq 0$ such that for any instant $t \geq 0$, we can find a large compact set Ω_t where the system states live in.

Before starting the analysis of the system observability along with the observer design, let us introduce the following definitions.

Definition 1: Consider the nonlinear system

$$
\begin{cases} \dot{x} = f(x, u),\\ y = h(x), \end{cases} \tag{2}
$$

where $x = x(t) \in \mathscr{M} \subset \mathbb{R}^n$ represents the system state vector, $f(\cdot, \cdot)$ is smooth vector with $f(0, 0) = 0$ and $u(t) \in \mathscr{U} \subset \mathbb{R}^m$ is the control input. The output nonlinearity $y = y(t) = h(x(t)) \in \mathbb{R}^p$ is supposed to be smooth with $h(0) = 0$. We say that system (2) is observable if for every two different initial conditions x_0 and $\bar{x}_0$ there exist an interval $[0,\, T]$, $T \in \mathbb{R}_{>0}$ and an admissible control $u(t)$ defined on $[0,\, T]$ such that the associated outputs $y(x_0, u(t))$, $y(\bar{x}_0, u(t))$ are not identically equal on $[0,\, T]$. We say, in this case, that the control input $u(t)$ distinguishes the pair $(x_0, \bar{x}_0)$ on $[0,\, T]$.

Definition 2: Consider system (2). The control input $u(t) \in \mathscr{U} \subset \mathbb{R}^m$ is said universal on $[0,\, T]$, if it distinguishes every different initial states $(x_0, \bar{x}_0)$ on $[0,\, T]$.

Definition 3: System (2) is said uniformly observable if every admissible control $u(t)$ defined on $[0,\, T]$, is a universal one.

Definition 4: System (2) is said to be algebraically observable if there exist two positive integers μ and ν such that

$$
x(t) = \phi\left(y,\, \dot{y},\, \ddot{y}, \cdots,\, y^{(\mu)},\, u,\, \dot{u},\, \ddot{u}, \cdots,\, u^{(\nu)}\right)(t), \tag{3}
$$

where $\phi(\cdot) : \mathbb{R}^{(\mu+1)p} \times \mathbb{R}^{(\nu+1)m} \mapsto \mathbb{R}^n$ is a differentiable vector valued nonlinearity that depends on the inputs, the outputs, and their respective higher derivatives.

According to the above definitions, system (1) is uniformly observable for any input. This property can be also checked via the algebraic observability of the system since the state vector is reconstructed by recurrence as follows:

$$
\begin{aligned}
x_1 &= y = \varphi_1(y),\\
x_2 &= \dot{y} - f_1(y, u) = \varphi_2(y, \dot{y}, u),\\
&\vdots\\
x_i &= \dot{\varphi}_{i-1}(y, \dot{y}, \cdots, y^{(i-2)}, u, \dot{u}, \cdots, u^{(i-3)})\\
&\quad - f_{i-1}\big(y, \varphi_2(y, \dot{y}, u), \cdots, \varphi_{i-1}(y, \dot{y}, \cdots, y^{(i-2)},\\
&\qquad\qquad u, \dot{u}, \cdots, u^{(i-3)}), u\big),\\
&= \varphi_i(y, \dot{y}, \cdots, y^{(i-1)}, u, \dot{u}, \cdots, u^{(i-2)}),\\
&\vdots\\
x_n &= \dot{\varphi}_{n-1}(y, \dot{y}, \cdots, y^{(n-2)}, u, \dot{u}, \cdots, u^{(n-3)})\\
&\quad - f_{n-1}\big(y, \varphi_2(y, \dot{y}, u), \cdots, \varphi_{n-1}(y, \dot{y}, \cdots, y^{(n-2)},\\
&\qquad\qquad u, \dot{u}, \cdots, u^{(n-3)}), u\big),\\
&= \varphi_n\big(y, \dot{y}, \cdots, y^{(n-1)}, u, \dot{u}, \cdots, u^{(n-2)}\big).
\end{aligned}
\tag{4}
$$

Since all the nonlinearities $f_i(x_1, \cdots, x_i)$, $1 \leq i \leq n$ are continuously differentiable then, the resulting functions $\varphi_i(s)$, $1 \leq i \leq n$ are well-defined with respect to their variables; hence, the states can be reconstructed from the information of the inputs and output without any singularity.

III. Observer analysis

A. n-th order time-varying differentiator

In this section, we present an adaptive-like differentiator whose successive states converge asymptotically to the successive higher derivatives of the input signal y. We show that the convergence of the differentiator is always assured even when y is not bounded. Before presenting this result let us introduce the following Lemma.