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Ergodic Control Problems for Optimal Stochastic Production Planning with Production Constraints

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Abstract—We study the ergodic control problem related to stochastic production planning in a single product manufacturing system with production constraints. The existence of a solution to the corresponding Hamilton-Jacobi-Bellman equation and its properties are shown. Furthermore, the optimal control for the ergodic control problem and an example are given.

Keywords—Single product manufacturing system, Stochastic production planning, Ergodic control problem, Hamilton-Jacobi-Bellman equation.

1. INTRODUCTION

This paper deals with the following first-order nonlinear differential equation:

$$\lambda = F\left(\frac{\partial v}{\partial x}(x,i)\right) - i\frac{\partial v}{\partial x}(x,i) + Av(x,i) + h(x), \qquad x \in R^1, \quad i = 1, 2, \dots, d. \tag{1}$$

Here, λ is a constant, $F(x) = kx$ if $x < 0$, $= 0$ if $x \geq 0$ for some positive constant $k > 0$, h is a convex function, and A denotes the infinitesimal generator of an irreducible Markov chain $(z(t), P)$ with state space $Z = \{1, 2, \dots, d\}$,

$$Av(x,i) = \sum_{j \neq i} q_{ij}[v(x,j) - v(x,i)], \tag{2}$$

where q_{ij} is the jump rate of $z(t)$ from state i to state j. The unknown is the pair (v, λ), where $v(\cdot, i) \in C^1(R^1)$ for every $i \in Z$.

Equation (1) arises in the ergodic control problem of stochastic production planning in a single product manufacturing system and is called the Hamilton-Jacobi-Bellman (HJB) equation or dynamic programming equation. The inventory level $x(t)$ of stochastic production planning modelled by Sethi and Zhang [1, Section 3.5, p. 50] is governed by the differential equation

$$\frac{dx(t)}{dt} = p(t) - z(t), \qquad x(0) = x, \quad z(0) = i, \quad P\text{-a.s.}, \tag{3}$$

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for production rate $0 \le p(t) \le k$, in which $z(t)$ and k are interpreted as the demand rate and capacity level, respectively. For the ergodic control problem, the cost $J(p(\cdot) : x, i)$ associated with $p(\cdot)$ is given by

$$J(p(\cdot) : x,i) = \limsup_{T\to\infty} \frac{1}{T} E\left[\int_0^T h(x(t))\,dt \mid x(0)=x,\, z(0)=i\right], \tag{4}$$

where $h(x)$ represents the convex inventory cost.

The purpose of this paper is to show the existence of a solution to the HJB equation (1) and to present the optimal control minimizing the cost $J(p(\cdot) : x,i)$ subject to (3). In the control problem of stochastic production planning with discounted rate $\alpha > 0$, Bensoussan *et al.* [2], Fleming *et al.* [3], and Sethi *et al.* [4] have investigated the HJB equation

$$\alpha u_\alpha(x,i) = F\left(\frac{\partial u_\alpha}{\partial x}(x,i)\right) - i\frac{\partial u_\alpha}{\partial x}(x,i) + Au_\alpha(x,i) + h(x). \tag{5}$$

For the ergodic control problem, we study the limit of (5) as α tends to 0, and investigate a solution to the degenerate HJB equation (1). This approach develops the technique of Bensoussan and Frehse [5] concerning nondegenerate second-order partial differential equations to our degenerate case.

Section 2 is devoted to the existence problem of the HJB equation (1) under the convexity assumption and others on h, and properties of the solution are shown in Section 3. In Section 4, an optimal control for the ergodic control problem and the value are given. In Section 5, we present an example of the solution to the HJB equation (1), and the optimal control and the value are given.

2. EXISTENCE

We are concerned with the equation

$$\alpha u_\alpha(x,i) = F\left(\frac{\partial u_\alpha}{\partial x}(x,i)\right) - i\frac{\partial u_\alpha}{\partial x}(x,i) + Au_\alpha(x,i) + h(x), \qquad x\in R^1, \quad i \in Z, \tag{6}$$

and make the following assumptions:

$$h(x) \text{ is nonnegative and convex on } R^1, \tag{7}$$

$$\exists C>0;\ h(x) \le C(1+|x|^\kappa) \text{ for some positive integer } \kappa, \tag{8}$$

$$k-d>0. \tag{9}$$

THEOREM 2.1. *We assume (7)–(9). Then, there exists a unique convex solution $u_\alpha(\cdot,i) \in C^1(R^1)$, $i \in Z$, of equation (6) such that*

$$\alpha\|u_\alpha(\cdot,i)\|_{L^\infty(I_r)} \le K_r, \tag{10}$$

$$\left\|\frac{\partial u_\alpha}{\partial x}(\cdot,i)\right\|_{L^\infty(I_r)} \le K_r, \tag{11}$$

$$\|Au_\alpha(\cdot,i)\|_{L^\infty(I_r)} \le K_r, \qquad i\in Z, \tag{12}$$

where K_r is a positive constant depending only on r of $I_r = (-r,r)$.

PROOF. According to Sethi and Zhang [1, Theorem 3.1, p. 40], equation (6) has a viscosity solution [6, Definition 4.1, p. 64] given by

$$u_\alpha(x,i) = \inf_{p(\cdot)\in\mathcal{P}(x,i)} \left\{E\left[\int_0^\infty e^{-\alpha t} h(x(t))\,dt \mid x(0)=x,\, z(0)=i\right]\right\},$$