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Necessary Conditions for 2D Systems' Stability ⋆

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Abstract: In this paper we consider parameterized Lyapunov inequalities arising in stability analysis of discrete-discrete and continuous-discrete 2D systems. A necessary condition for their feasibility is presented. This condition is based on an optimization method published earlier by the author. Its properties include being reasonably easy to check, and its coverage being adjustable via the algorithm's settings.

Keywords: 2D systems, global optimization, Lyapunov method, nonlinear programming, matrix inequalities, polynomial inequalities.

1. INTRODUCTION

A number of control problem areas including, but not limited to, analysis of repetitive processes (see Owens and Rogers (1995); Rogers and Owens (1992); Rogers et al. (2007)), distributed parameter systems (see Dullerud and D'Andrea (2004); Rabenstein and Trautmann (2003); Cichy et al. (2008)), disturbance propagation (see Knorn and Middleton (2013); Li et al. (2005)), employ the formalism of multidimensional (nD) systems. The most frequently considered case is $n = 2$ with both independent variables having some variation of temporal semantics: for example, one variable can serve as a continuous time parameter with a limited span, and another—a discrete one—being an iteration number. Representations with both time variables being discrete are also well known.

Analysis of such systems involves questions not unlike the ones already answered in the classical control theory, however, applying this theory here is a significantly harder task. Even the basic question of linear system stability requires nontrivial tests and computations.

As with the more usual kinds of dynamic systems, there are various approaches to 2D systems' stability analysis. They are generally based on characteristic polynomials/multinomials (see, e.g., Agathoklis et al. (1993) for the double-discrete case, and Rogers and Owens (2002) for continuous-discrete time systems), or specialized Lyapunov functions. The latter often allows using techniques based on linear matrix inequalities (LMIs) and convex optimization. It also supports more complex types of analysis/synthesis problems (see, e.g., Paszke et al. (2011)).

The current paper utilizes the Lyapunov-function-based approach which leads to new problem statements in terms of linear or polynomial matrix inequalities (LMIs or PMIs). It should be noted, however, that even when respective results are necessary and sufficient (which is not always the case; see, e.g., Anderson et al. (1986)), doing full tests is not guaranteed to be practical due to complexities of solving PMIs. At the same time, limiting tests to partial ones by, for example, replacing PMIs with their approximations, turns necessary and sufficient conditions into merely sufficient ones. This matter, and a way to make the issue somewhat less pronounced, is further discussed in section 2.2.

This paper proposes a sufficient condition for infeasibility—or, equivalently, a necessary conditions for feasibility—of parameterized Lyapunov inequalities for mixed continuous-discrete-time and discrete-discrete systems.

Section 2 provides the problem statement and relevant background information from existing works. The main contribution of this paper is presented in section 3; section 4 shows a numerical example.

2. PRELIMINARIES

2.1 2D Systems

2D systems have a number of representations, the most well-known of which are the following ones.

The Roesser model (Roesser (1975)) is given by:

$$\begin{bmatrix} x_h(i+1,j) \\ x_v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_{hh} & A_{hv} \\ A_{vh} & A_{vv} \end{bmatrix} \begin{bmatrix} x_h(i,j) \\ x_v(i,j) \end{bmatrix} + \begin{bmatrix} B_h \\ B_v \end{bmatrix} u(i,j),$$
$$y(i,j) = \begin{bmatrix} C_h & C_v \end{bmatrix} \begin{bmatrix} x_h(i,j) \\ x_v(i,j) \end{bmatrix} + Du(i,j), \tag{1}$$

where $i \in \mathbb{N}_0$ and $j \in \mathbb{N}_0$ are discrete time variables, $x_h \in \mathbb{R}^{n_h}$ and $x_v \in \mathbb{R}^{n_v}$ are the "horizontal" and "vertical" state components, $u \in \mathbb{R}^{n_u}$ is the vector of input variables, $y \in \mathbb{R}^{n_y}$ is the vector of output variables, A_{**}, B_*, C_*, and D are appropriately sized given matrices. Boundary conditions in their simplest form are given by

$$x_h(0,j) = x^h_{0j} \in \mathbb{R}^{n_h},\ x_v(i,0) = x^v_{i0} \in \mathbb{R}^{n_v},\ i,j \in \mathbb{N}_0.$$

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A discrete-discrete 2D repetitive process is similar to the Roesser model (Rogers et al. (2007)):

$$x_{k+1}(p+1) = Ax_{k+1}(p) + B_0 y_k(p) + Bu_{k+1}(p),$$
$$y_{k+1}(p) = Cx_{k+1}(p) + D_0 y_k(p) + Du_{k+1}(p),$$

where $k \in \mathbb{N}_0$ and $p \in \mathbb{N}_0$ are discrete time variables with k being the iteration number, and $p < \alpha < \infty$ being the discretized time for the current iteration, α is a given constant, and the rest of notation keeping their general meaning. The simplest form of boundary conditions is

$$x_{k+1}(0) = d_{k+1} \in \mathbb{R}^n,\ k \geq 0,\ y_0(p) = f(p) \in \mathbb{R}^m,$$

where $f(p)$ is a given function.

A 2D mixed continuous-discrete-time repetitive process can be represented as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\,t} x_c(t,k) &= A_{cc} x_c(t,k) + A_{cd} x_d(t,k) + B_c u(t,k), \\ x_d(t,k+1) &= A_{dc} x_c(t,k) + A_{dd} x_d(t,k) + B_d u(t,k), \\ y(t,k) &= C_c x_c(t,k) + C_d x_d(t,k) + Du(t,k), \end{aligned} \tag{2}$$

where $t \in \mathbb{R}$ and $k \in \mathbb{N}_0$ are the continuous and discrete time variables, $x_c \in \mathbb{R}^{n_c}$ and $x_d \in \mathbb{R}^{n_d}$ are the continuous and discrete states, and the rest of notation keep their general meaning.

In this paper, we will consider the problem of stability of the discrete-discrete Roesser form (other forms can be analyzed in similar ways, see, e.g., Hinamoto (1993)) and the mixed continuous-discrete model.

Paper Chesi and Middleton (2014) shows that exponential stability of (2), defined as existence of $\beta, \gamma \in \mathbb{R}$, $\beta > 0$, $\gamma > 0$, such that

$$\left\| \begin{pmatrix} x_c(t,k) \\ x_d(t,k) \end{pmatrix} \right\|_2 \leq \beta\rho \mathrm{e}^{-\gamma \min\{t,k\}},$$
$$\rho = \max\{\sup_{t\geq 0} \|x_d(t,0)\|_2, \sup_{k \geq 0} \|x_c(0,k)\|_2\},$$

for all initial conditions $x_c(0,k) \in \mathbb{R}^{n_c}$ and $x_d(t,0) \in \mathbb{R}^{n_d}$ for all $t \geq 0$ and $k \geq 0$, is equivalent to feasibility of the problem

$$\begin{aligned} &\forall \omega \in \mathbb{R}: \\ &P(\omega) \geq cI, \\ &P(\omega) - F(j\omega)^H P(\omega) F(j\omega) \geq cI, \\ &c > 0, \end{aligned} \tag{3}$$

where $P(\omega) \in \mathbb{C}^{n_d \times n_d}$ is a Hermitian matrix function (that can be chosen as a polynomial of degree no greater than $2n_c n_d^2$), and $F(s) = A_{dc}(sI - A_{cc})^{-1} A_{cd} + A_{dd}$. Authors provide two necessary and sufficient conditions which boil down to solving certain LMI problems. These conditions, however, are only practical for relatively low dimensions of state spaces involved: for example, their numbers of LMI variables grow as $O(n_c^2 n_d^4)$ and up to $O(n_c^2 n_d^6)$, respectively.

For discrete-discrete systems, a similar result is known from Agathoklis et al. (1993). The system (1) is stable iff absolute values of A_{vv}'s eigenvalues are less than 1, and the "frequency-dependent" Lyapunov equation

$$F(\mathrm{e}^{j\omega})^H P(\mathrm{e}^{j\omega}) F(\mathrm{e}^{j\omega}) - P(\mathrm{e}^{j\omega}) = -Q(\mathrm{e}^{j\omega}) \tag{4}$$

with $F(s) = A_{hh} + A_{hv}(\mathrm{e}^{j\omega} I - A_{vv})^{-1} A_{vh}$ has a Hermitian positive definite solution $P(\mathrm{e}^{j\omega})$ for all $\omega \in [0, 2\pi]$ and all Hermitian positive definite matrices $Q(\mathrm{e}^{j\omega})$. Authors provide this problem's feasibility condition in terms of certain matrix' eigenvalues. The condition is not quite constructive, but the equation is still interesting by itself.

We can see that both cases are essentially parameterized discrete Lyapunov inequalities that need to be satisfied for all values of the parameter in order for the dynamic system to be stable.

2.2 Atomic Optimization

Consider the following problem:

$$\begin{aligned} &f^* = \min_x f(x), \\ &G_i(x) \geq 0, \\ &x \in \mathbb{R}^n, \quad G_i(x) = G_i^{\mathrm{T}}(x) \in \mathbb{R}^{n_i \times n_i}, \quad i = 1, \dots, m, \end{aligned} \tag{5}$$

where $f(x)$ and elements of $G_i(x)$ are (not necessarily convex) polynomials, and the inequality sign in (5) denotes positive semidefiniteness. Hereafter, these problems will be called *PMI problems*.

In Lasserre (2001); Henrion and Lasserre (2005, 2006), a solution method for this kind of problems has been proposed. It was based on constructing a hierarchy of LMI "relaxations" that would approximate the original problem in the space of its variables' moments. This powerful global optimization method did, nonetheless, suffer from combinatorial explosion of LMI relaxation sizes. Papers Pozdyayev (2013, 2014) present a transformation of this method aimed at significantly reducing its computational complexity while maintaining its key benefits for a class of problems related to control theory, in particular, to the Lyapunov method.

The optimization method presented in the last two papers—dubbed "atomic optimization"—combines a simple interior point method of solving LMI problems with non-LMI representations of LMI relaxations. These non-LMI representations can be equivalent to LMI relaxations, or, unlike the latter, they can be significantly reduced in size while staying compatible with the general solution method. Depending on the reduction level, the resulting algorithm can vary in power, ranging from local search to global search.

For (5), the transformed problem proposed in Pozdyayev (2014) is constructed as follows. Here, we provide a brief recap of relevant equations; for details and variations, see the original paper.

Let $r \geq 1$ be the number of atoms, $x_i \in \mathbb{R}^n$, $i = 1, \dots, r$, be their (unknown) values, and $p_i \in \mathbb{R}$, $\sum_{j=1}^r p_j = 1$, their (unknown) weights; we will denote the combined vector of variables as $z = [x_{11}\ x_{12}\ \dots\ x_{1n}\ x_{21}\ \dots\ x_{rn}\ p_1\ p_2\ \dots\ p_r]$. Let $V \in \mathbb{R}^{(n+1)\times r}$ be the n-D Vandermonde matrix of order 1 for vectors x_i:

$$V = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_{11} & x_{21} & \dots & x_{r1} \\ x_{12} & x_{22} & \dots & x_{r2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \dots & x_{rn} \end{bmatrix}.$$

The simplest form of the transformed problem derived directly from the first order relaxation with $r = n + 1$ is