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Robust PID controller design for processes with stochastic parametric uncertainties

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A B S T R A C T

The stability and performance of a system can be inferred from the evolution of statistical characteristics of the system's states. Wiener's polynomial chaos can provide an efficient framework for the statistical analysis of dynamical systems, computationally far superior to Monte Carlo simulations. This work proposes a new method of robust PID controller design based on polynomial chaos for processes with stochastic parametric uncertainties. The proposed method can greatly reduce computation time and can also efficiently handle both nominal and robust performance against stochastic uncertainties by solving a simple optimization problem. Simulation comparison with other methods demonstrated the effectiveness of the proposed design method.

1. Introduction

Most engineering applications involve solving physical problems by converting them into deterministic mathematical models with fixed physical parameters. In reality however, these parameters show some randomness, which influences the behavior of the solution. This randomness is not incorporated into deterministic models. Several probabilistic methods have been developed to include this uncertainty in mathematical models.

A representative traditional probabilistic approach for uncertainty quantification is the Monte Carlo (MC) method [1,2]. Its brute-force implementation involves first generating an ensemble of random realizations with each parameter randomly drawn from its uncertainty distribution. Deterministic solvers are then applied to each member to obtain an ensemble of results. The ensemble of results is then post-processed to estimate relevant statistical properties, such as the mean and standard deviation. The estimation of the mean converges with the inverse square root of the number of runs, making MC methods computationally expensive.

Polynomial chaos (PC) is a modern approach to quantifying uncertainty in system models. A PC expansion, originating from Wiener chaos [3], is a spectral representation of a random process by orthonormal polynomials of a random variable. For PC expansions of infinite smooth functions (i.e., analytic, infinitely differentiable), exponential convergence is expected. Ghanem and Spanos [4] showed that PC is an effective computational tool for engineering purposes. Karniadakis and Xiu [5] further generalized PC for use with non-standard distributions. Puvkov et al. [6] proposed that if the Wiener–Askey polynomial chaos expansion is chosen according to the probability distribution of the random input, then it can be used to construct simple algorithms for the statistical analysis of dynamic systems. Several PC expansion-based methods have been proposed including non-intrusive polynomial chaos expansions [7], the stochastic response surface method [8], and the deterministic equivalent modeling method (DEMM) [6,9,10], also known as the probabilistic collocation method (PCM). Controllers for processes with stochastic uncertainties can be designed based on the statistical characteristics of the response. Each distribution has an associated optimal polynomial, which gives optimal convergence. Given that the PCM can compute response statistics effectively, bi-level approaches to controller design under uncertainty have been pursued [6].

This work proposes an efficient method for robust PID controller design that uses the PCM for processes with stochastic uncertainties. The proposed method greatly reduces computational time for controller design and also efficiently handles both nominal and robust performance against uncertainties.

This paper proceeds as follows: Section 2 briefly introduces the basic concepts and theory of the PC method. The PC method is then applied to the statistical analysis of interval systems and robust PID controller design. In Section 3, robust PID controllers are designed for several representative systems with stochastic uncertainties by judging the statistical characteristics of the response. Simulation study compares the proposed PID controller with other methods.

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2. Statistical analysis using polynomial chaos theory

2.1. Generalized polynomial chaos (gPC) theory

Consider a control system governed by differential algebraic equations [10]:

$$\begin{cases} F(t, y, y', \ldots, y^{(l)}, \xi) = 0 \\ g(t_0, y(t_0), \ldots, y^{(l)}(t_0), \xi) = 0 \end{cases} \quad (1)$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$ is a random vector of mutually independent random components with probability density functions of $\rho_i(\xi_i) : \Gamma_i \to \mathbb{R}^+$; y denotes a state variable.

Thus, the joint probability density of the random vector, ξ, is $\boldsymbol{\rho} = \prod_{i=1}^{N} \rho_i$, and the support of ξ is $\boldsymbol{\Gamma} \equiv \prod_{i=1}^{N} \Gamma_i \in \mathbb{R}^N$. The set of one-dimensional orthonormal polynomials, $\{\phi_i(\xi_i)_{m=0}^{d_i}\}$, can be defined in the finite dimension space, Γ_i, with respect to the weight, $\rho_i(\xi)$. Based on the one-dimensional set of polynomials, an **N**-variate orthonormal set can be constructed with P total degrees in the space $\boldsymbol{\Gamma}$ by using the tensor product of the one-dimensional polynomials, the basis function of which satisfies:

$$\int_{\Gamma} \Phi_m(\xi)\Phi_n(\xi)\boldsymbol{\rho}(\xi)d\xi = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \quad (2)$$

Considering a response function $f(y(t, \xi))$, with statistics (e.g., mean, variance) of interest; the **N**-variate Pth order approximation of the response function can be constructed as:

$$f_N^P(y(t,\xi)) = \sum_{i=1}^{M} \widehat{f}_i(t)\Phi_i(\xi); \qquad M + 1 = \begin{pmatrix} N+P \\ N \end{pmatrix} = \frac{(N+P)!}{N!P!} \quad (3)$$

where P is the order of polynomial chaos, and $\widehat{f}_m$ the coefficient of the gPC expansion that satisfies (2) as:

$$\widehat{f}_i = \mathbf{E}[\Phi_i f(y)] = \int_{\Gamma} f(y)\Phi_i(\xi)\rho(\xi)d\xi \quad (4)$$

where **E**[] denotes the expectation operator.

2.2. Probabilistic collocation for statistical analysis of control systems

In this work, a probabilistic collocation approach [10] is employed for the statistical analysis of control systems to take advantage of its capability to deal with complex response functions effectively. Its algorithm is briefly:

- Choose a collocation set, $\{\xi_i^{(m)}, w_i^{(m)}\}_{m=1}^{q_i}$, for each random component, ξ_i, for every direction $i = 1, \ldots, N$, and construct a one-dimensional integration rule:

$$Q_{q_i}^{(i)}[g] = \sum_{j=1}^{q_i} g(\xi_i^{(j)})w_i^{(j)} \quad (5)$$

  where $Q[]$ denotes the quadrature approximation of the univariate integration, $g()$ is a response function of system state, statistics of which are needed to estimate; $w_i^{(j)}$ and $\xi_i^{(j)}$ are the jth weight and node taken from the collocation set for random component ξ_i. A Gaussian quadrature (or a Gaussian collocation set) [11] is usually used as the one-dimensional integral rule in classical spectral methods such as the PCM (DEMM).
- Obtain an **N**-dimensional integration rule (cubature nodes and weights) by the tensorization of the one-dimensional integral rule:

$$\ell^Q[g] = (Q_{q_1}^{(1)} \otimes \cdots \otimes Q_{q_N}^{(N)})[g] = \sum_{j_1=1}^{q_1} \cdots \sum_{j_N=1}^{q_N} g(\xi_j^{(j_1)}, \ldots, \xi_j^{(j_N)})(w_1^{(j_1)} \cdots w_1^{(j_N)}) \simeq \int_{\Gamma} g(\xi)\rho(\xi)d\xi \quad (6)$$

  where $\otimes$ denotes the tensor product, and $\ell^Q[]$ denotes the multivariate cubature approximation.
- Approximate the gPC coefficients in (4) using the numerical integration rule in (6).

$$\widehat{f}_j = \ell^Q[f(y,\xi)\Phi(\xi)\boldsymbol{\rho}] = \sum_{m=1}^{Q} f(\xi^{(m)})\Phi_j(\xi^{(m)})w^{(m)} \quad \text{for} \quad j = 1, \ldots, M \quad (7)$$

  where $\widehat{f}$ represents the numerical approximation of $\widehat{f}$ and $f(\xi)\Phi_j(\xi)$ play the role of $g(\xi)$ in Eq. (6).
- Construct an **N**-variate Pth order gPC approximation of the response function in the form: $\tilde{f}_N^P = \sum_{j=1}^{M} \widehat{f}_j \Phi_j(\xi)$.

Once all the gPC coefficients have been evaluated, a post-processing procedure is then carried out to obtain the statistics of the response function $f(y(t, \xi))$.

The mean value is the first expansion coefficient:

$$\mathbf{E}[\tilde{f}_N^P] = \mu_f = \int_{\Gamma} \tilde{f}_N^P \rho(\xi)d\xi = \int_{\Gamma} \left[\sum_{j=1}^{M} \widehat{f}_j(\xi)\Phi_j(\xi) \right] \rho(\xi)d\xi = \widehat{f}_1 \quad (8)$$

The variance of the response function $f(y(t, \xi))$ can be evaluated as:

$$D_f = \sigma_f^2 = \mathbf{E}[(f - \mu_f)^2] = \int_{\Gamma} \left(\sum_{j=1}^{M} \widehat{f}_j(\xi)\Phi_j(\xi) - \widehat{f}_1 \right) \times \left(\sum_{j=1}^{M} \widehat{f}_j(\xi)\Phi_j(\xi) - \widehat{f}_1 \right) \rho(\xi)d\xi = \sum_{j=2}^{M} \widehat{f}_j^2 \quad (9)$$

Eqs. (8) and (9) employ the property that the polynomial set begins with $\Phi_1(\xi) = 1$. The weight function of the polynomial is the probability density function. If an $f(y) = y$ response function is chosen, the mean and variance of the system's states are approximately given by (8) and (9), respectively. The surrogate gPC series $\tilde{f}_N^P = \sum_{j=1}^{M} \widehat{f}_j \Phi_j(\xi)$ can be sampled for obtaining the probability density function for the response function.

The set $\{\phi_i\}_{i=1}^{d_i}$ is the orthonormal polynomial of ξ_i with weight function $\rho_i(\xi_i)$. The weight function is the probability density function of random variable ξ_i. This links the distribution of random variable ξ_i and the type of the orthonormal polynomial in its gPC basis.

2.3. Polynomial chaos for arbitrary distribution and its associate quadrature

Let $\rho(\xi)$ be the probability density function of a scalar random variable, ξ, which has finite moments of order up to $2m$, $m \in \mathbb{N}$. Let P denote the space of a real polynomial, with $P_m \subset P$ denoting the space of the polynomial with degree up to m. The inner product of two polynomials, p and q, relative to $d\lambda = \rho(\xi)d\xi$ is defined by:

$$(p, q)_{d\lambda} = \int_{\Gamma} p(\xi)q(\xi)\rho(\xi)d\xi \quad (10)$$