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Production planning and inventories optimization: A backward approach in the convex storage cost case

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Abstract

We study the deterministic optimization problem of a profit-maximizing firm which plans its sales/production schedule. The firm controls both its production and sales rates and knows the revenue associated to a given level of sales, as well as its production and storage costs. The revenue and the production cost are assumed to be respectively concave and convex. In Chazal et al. [Chazal, M., Jouini, E., Tahraoui, R., 2003. Production planning and inventories optimization with a general storage cost function. Nonlinear Analysis 54, 1365–1395], we provide an existence result and derive some necessary conditions of optimality. Here, we further assume that the storage cost is convex. This allows us to relate the optimal planning problem to the study of a backward integro-differential equation, from which we obtain an explicit construction of the optimal plan.

JEL classification: D21; D42

Keywords: Production planning; Inventory management; Integro-differential equations

1. Introduction

We consider a firm which produces and sells a good which can be stored. The firm acts in continuous time on a finite period in order to maximize dynamically its profit. Its instantaneous profit is given by the revenue rate entailed by the sales, diminished by the production cost rate, and by the cost of storage of the current inventories. Our approach to this production planning and inventory management problem is in the same vein as the one launched by Arrow et al. (1958).

Many contributions to this theory have been brought from the 1950s until now, with different approaches. A considerable part of the literature is devoted to the study of cost-minimizing firms. Basically, in production-planning models, the firm makes decisions in order to minimize its production and storage costs and on the basis of the good demand forecast. In production-smoothing models, which were initiated by Holt et al. (1960), additional adjustment

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cost are taken into account. Some alternative to these models is given by the (S, s) models of inventory behavior. See notably Scarf (1960) for a basic existence result. A unified mathematical treatment of production-planning and production-smoothing problems was proposed by Bensoussan et al. (1983). Applications of optimal control theory in management science and in particular in production planning can be found in Sethi and Thompson (1981) or in Maimon et al. (1998), for more recent results.

All models of this prominent part of the production-planning literature assume that the sales of the firm are completely driven by the possibly time-varying and stochastic demand. In this paper, we work in a deterministic context: the firm knows the revenue associated to a certain level of sales as well as the cost of production and the cost of storage. In addition, we assume that the firm controls not only its production rate but also its sales rate. When the demand of the product is a bijective function of the price, this amounts to assume that the firm chooses the price. This is typically the case of a monopolistic firm. Our model is closely related to the one by Pekelman (1974), who was probably the first to take price as a decision variable. Feichtinger and Hartl (1985) extended Pekelman's work to the case of a general convex storage cost function (not necessarily linear) as well as to the case of a nonlinear demand function. Following these works, we will assume that the production and storage costs are given by some general convex functions. To the contrary, we will not impose the revenue function to be computed as the price times the demand function. It will be given by some general concave function of the sales rate. Another main difference with Feichtinger and Hartl's model is that we do not allow for backlogging.

The sales/production-planning problem of the profit-maximizing firm is formulated as an optimal control problem where the controls, namely the sales and production paths, are integrable. In other words, the cumulative production and sales processes are assumed to be absolutely continuous. The investigation of the problem is not straightforward due to the presence of control and state constraints. It has been studied by the same authors in Chazal et al. (2003) with a general storage cost function. We proved an existence result for a relaxed problem in which the cumulative sales process is allowed to have a jump at time 0 (i.e. the firm is allowed to make a partial depletion of its eventual initial inventory at time 0). We also derived the first order conditions of optimality for both problems and thus provided a qualitative description of the optimal plans. In particular, we establish the following result: the optimal inventory level must decrease until it reaches 0. It is then kept null by producing for immediate selling.

In the present article, we go further in this analysis. For this purpose, we assume that the storage cost is convex. In this context, we first see that the initial problem has a solution if and only if the relaxed one has a solution without jump at time 0. Second, the first order conditions of Chazal et al. (2003) allow us to characterize the (unique) optimal plan for the relaxed problem. Using this characterization, we relate the relaxed optimization problem to the study of a class of solutions of a backward integro-differential equation, indexed by a class of admissible terminal conditions. Using some standard existence result for retarded equations given in Hale and Verduyen Lunel (1993), we establish existence and uniqueness of a maximal solution for this backward integro-differential equation. We then study the behavior of the solution with respect to the terminal condition. This allows us to provide a constructive description of the optimal plan. The optimal plan is determined by selecting the greatest terminal condition r such that a certain functional of the solution of the backward integro-differential equation, representing the inventory level at time 0+, $S_0(r)$, remains lower than the exogenous initial inventory s_0. The difference $\alpha = s_0 - S_0(r)$ corresponds to the size of the jump of the cumulative sales process at time 0. After 0, the sales and production rates are explicitly determined in function of the corresponding maximal solution.

We obtain that there exists an exogenous threshold $\bar{s}_0$ on the initial inventory above which the size of the jump, α, is positive. The economic interpretation is the following. If the initial inventory s_0 entails too high storage costs, i.e. s_0 is greater than $\bar{s}_0$, then it is optimal for the firm to sell out immediately the quantity $s_0 - \bar{s}_0$, so as to reduce its initial inventory to $\bar{s}_0$. If the initial inventory is lower than $\bar{s}_0$, then the firm has no interest in selling out immediately. The level $\bar{s}_0$ appears as the maximal level that it can afford to hold. We also obtain that this threshold depends on the length of the planning period and we observe some qualitative differences between the optimal sales/production plans obtained on the short planning periods and on the long ones.

The paper is organized as follows. Section 2 provides a precise description of the model and recalls some basic results of Chazal et al. (2003). Section 3 is devoted to the backward characterization of the optimal plan for the relaxed problem. The constructive resolution of the production-planning problem is given in Section 4. The proofs of our main results are given in Section 5 by the way of the study of the above mentioned backward integro-differential equation.