Applied Mathematics and Computation 204 (2008) 568–579

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Direct optimization methods for solving a complex state-constrained optimal control problem in microeconomics

Helmut Maurer [a,*], Hans Josef Pesch [b]

[a] Universität Münster, Institut für Numerische und Angewandte Mathematik, Einsteinstrasse 62, 48149 Münster, Germany
[b] Universität Bayreuth, Fakultät für Mathematik und Physik, Universitätstrasse 30, 95440 Bayreuth, Germany

A R T I C L E I N F O

Keywords:
Microeconomic control model
Control of stock
Labor and capital
State inequality constraints
Direct optimization methods
Bang–bang and singular control
Verification of necessary and sufficient conditions

A B S T R A C T

We analyze and solve a complex optimal control problem in microeconomics which has been investigated earlier in the literature. The complexity of the control problem originates from four control variables appearing linearly in the dynamics and several state inequality constraints. Thus, the control problem offers a considerable challenge to the numerical analyst. We implement a hybrid optimization approach which combines two direct optimization methods. The first step consists in solving the discretized control problem by nonlinear programming methods. The second step is a refinement step where, in addition to the discretized control and state variables, the junction times between bang–bang, singular and boundary subarcs are optimized. The computed solutions are shown to satisfy precisely the necessary optimality conditions of the Maximum Principle where the state constraints are directly adjoined to the Hamiltonian. Despite the complexity of the control structure, we are able to verify sufficient optimality conditions which are based on the concavity of the maximized Hamiltonian.

1. Introduction

The well-known microeconomic concern model of Lesourne and Leban [10] involves only capital flows as control and state variables. Koslik and Breitner [8] and Winderl and Naumer [16] have developed an extended concern model that includes the production and employment sector. Besides its economic interest, the optimal control problem constitutes a considerable numerical challenge, since it comprises four control variables appearing linearly in the dynamics and several pure state inequality constraints.

In [8,16], a hybrid numerical approach has been developed to determine the complicated control switching structure. First, a discretized version of the control problem is solved by nonlinear programming methods. This method yields reliable estimates for the control and state variables on a fixed grid. The second step is a refinement step, where the control and state estimates are used in the so-called indirect method which requires the solution of a boundary value problem (BVP) for the state and adjoint variables. For the concern model, it is extremely difficult to set up the BVP due to the presence of pure state constraints. For this reason, authors [8,16] have substituted the active state constraints by suitable mixed control-state constraints that are better tractable in the BVP formulation.

The purpose of the paper is twofold. First, we discuss direct optimization methods that provide solutions which satisfy precisely the Maximum Principle for state-constrained optimal control problems.

* Corresponding author.
E-mail addresses: maurer@math.uni-muenster.de (H. Maurer), hans-josef.pesch@uni-bayreuth.de (H.J. Pesch).

0096-3003/$ - see front matter © 2008 Published by Elsevier Inc.
doi:10.1016/j.amc.2008.05.035

The second goal is to show that the computed solution satisfies a suitable type of sufficient optimality conditions. The organisation of the paper is as follows. The concern model is presented in Section 2. In Section 3, necessary optimality conditions are discussed which are based on a Maximum Principle, where the state constraints are directly adjoined to the Hamiltonian. In Section 4, we present a hybrid optimization approach to solve the state-constrained control problem. The first step is similar to that in [8,16] and differs only in that we apply the large-scale optimization methods developed by Büskens [2], Büskens and Maurer [3] and Wächter and Biegler [15]. The second step is different from the one in [8,16]. Instead of trying to solve the BVP of the Maximum Principle, we optimize simultaneously the switching and junction times between bang–bang, singular and boundary arcs and the discretized control variables; cf. [5,13,14]. The computed control and state variables satisfy the Maximum Principle with high accuracy. Finally, in Section 5, we show that the computed solutions satisfy a suitable type of sufficient conditions.

2. Optimal control model for a concern with four control variables appearing linearly and state constraints

The microeconomic control model discussed in Koslik and Breitner [8] and Winderl and Naumer [16] has six state variables and four control variables:

$$x = (S, L, Y, X, X_{\mathrm{m}}, X_{\mathrm{r}}) \in \mathbb{R}^6, \quad u = (S_{\mathrm{c}}, L_{\mathrm{c}}, Y_{\mathrm{c}}, I) \in \mathbb{R}^4,$$

which have the following meaning. The stock $S(t)$ is controlled by $S_{\mathrm{c}}(t)$; the number $L(t)$ of employees is controlled by the employment rate $L_{\mathrm{c}}(t)$; the capital consists of loan capital $Y(t)$ and equity capital $X(t)$; the control $Y_{\mathrm{c}}(t)$ describes the borrowing of loan capital while the owners of the equity capital choose by means of the investment control $I(t)$ between an investment within the concern and an alternative investment $X_{\mathrm{m}}(t)$; the risk premium $X_{\mathrm{r}}(t)$ serves as a reserve fund for the safety of the capital owners; the risk premium is denoted by $\rho_{\mathrm{r}}(t)$. All parameters and functions appearing in the following quantities and differential equations are summarized in Table 1. The production function (output) is assumed to be of Cobb–Douglas type:

$$F(x) = F(L, Y, X) = \alpha (X + Y)^{\alpha_K} L^{\alpha_L}. \tag{1}$$

Then the profit (gain) of the concern is given by

$$G(x, u, t) = \frac{1}{d(t)} [p(t)(F(x) - S_{\mathrm{c}}) - \sigma S - \omega(t) L] - \rho_K(t) Y - \delta(X + Y). \tag{2}$$

The discount rate $d(t)$ is defined as the solution of the differential equation:

$$\dot{d}(t) = -d(t) \ln(1 + i(t)), \tag{3}$$

where $i(t)$ is the periodic inflation rate specified in Table 1. Note that in contrast to the presentation in [8,16], we do not treat $d(t)$ as a state variable. In Section 5, this viewpoint will allow us to apply sufficient optimality conditions.

The dynamics is governed by differential equations with fixed initial values,

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(0) = x_0, \tag{4}$$

Table 1
Parameter and function values for the microeconomic control model

Notation	Formula/value	Meaning
t_{f}	10	Time horizon in years
$F(x)$	$\alpha(X+Y)^{\alpha_K} L^{\alpha_L}$	Production function (output)
α	100	Parameter in production function
α_K	0.35	Elasticity of total capital $K = X + Y$
α_L	0.5	Elasticity of labor
k_l	8	Duration of economic cycle
$k_p(t)$	$\frac{\pi}{2} + \frac{2\pi}{k_l} \cdot t$	Position in economic cycle
$\rho_K(t)$	$0.110 + 0.030 \sin k_p(t)$	Loan interest rate
$\rho_{\mathrm{m}}(t)$	$0.074 + 0.018 \sin k_p(t)$	Current yield
$\rho_{\mathrm{r}}(t)$	$\rho_K(t) - 0.05$	Risk premium rate
$\rho_{\mathrm{r}}^{\mathrm{low}}(t)$	$\frac{2}{3}(\rho_K(t) - 0.08) + 0.02$	Risk premium rate for daring investors
$i(t)$	$0.019 + 0.029 \sin k_p(t)$	Inflation rate
p	0.05	Constant selling price
$p(t)$	$0.05 + 0.01 \sin\left(\frac{2\pi}{k_l} \cdot t\right)$	Variable selling price
κ	0.8	Rate of maximal borrowing
ω	2.0	Constant labor cost
$\omega(t)$	$2.0 \exp(0.02 \cdot t)$	Increasing labor cost
δ	0.440	Depreciation rate
τ	0.5	Tax rate
σ	0.01	Storage charges