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Achieving matrix consistency in AHP through linearization

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ABSTRACT

Matrices used in the analytic hierarchy process (AHP) compile expert knowledge as pairwise comparisons among various criteria and alternatives in decision-making problems. Many items are usually considered in the same comparison process and so judgment is not completely consistent – and sometimes the level of consistency may be unacceptable. Different methods have been used in the literature to achieve consistency for an inconsistent matrix. In this paper we use a linearization technique that provides the closest consistent matrix to a given inconsistent matrix using orthogonal projection in a linear space. As a result, consistency can be achieved in a closed form. This is simpler and cheaper than for methods relying on optimisation, which are iterative by nature. We apply the process to a real-world decision-making problem in an important industrial context, namely, management of water supply systems regarding leakage policies – an aspect of water management to which great sums of money are devoted every year worldwide.

1. Introduction

The analytic hierarchy process (AHP) [1] provides a useful method to establish relative scales that can be derived by making pairwise comparisons using numerical judgments from an absolute scale of numbers. This approach is essential, for example, when tangible and intangible factors need to be considered within the same pool. The various factors are arranged in a hierarchical or a network structure with the objective(s) at the top, followed by one or more layers of criteria, and finally, the alternatives at the bottom. The ability of alternatives to achieve the objective(s) is measured according to the criteria represented within the structure. To this end, the people involved in the process compare the criteria and the alternatives in pairs, make judgments, and compile the results into matrices (matrices of criteria or matrices of alternatives).

Any two elements, for example criteria C_i and C_j are semantically compared. A value a_{ij} is proposed directly (numerically) or indirectly (verbally) that represents the judgment of the relative importance of the decision element C_i over C_j. Among the different approaches for developing such scales [2] the nine-point scale developed by Saaty [3] is one of the most popular. By using the Saaty scale, if the elements C_i and C_j are considered to be equally important, then $a_{ij} = 1$ (homogeneity). If C_i is preferred to C_j, then $a_{ij} > 1$, with an integer grade ranging from 2 to 9 that respectively corresponds to weak, moderate, ..., until very strong, and extreme importance of C_i over C_j. Intermediate numerical (decimal) values in the scale may be used to model hesitation between two adjacent judgments [1,4,5]. It is assumed that the reciprocal property $a_{ji} = 1/a_{ij}$ always holds. Homogeneity also implies that $a_{ii} = 1$ for all $i = 1, 2, \dots, n$. In this way, a homogeneous and reciprocal $n \times n$ matrix of pairwise comparisons A is compiled. This approach is intended to embody expert know-how regarding a specific problem. Matrices such as A are positive matrices (matrices with only positive entries) that also exhibit homogeneity and reciprocity.

There are different techniques to extract priority vectors from these comparison matrices [6–8]. The eigenvector method, proposed by Saaty in his seminal paper [3] in 1977, stands out from the rest. Saaty proved that the Perron eigenvector of the

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comparison matrix provides the necessary information to deal with complex decisions that involve dependence and feedback - as analyzed in the context of, for example, benefits, opportunities, costs, and risks [9]. The required condition is that the matrix exhibits a minimum level of consistency. Consistency expresses the coherence that should (perhaps) exist between judgments about the elements of a set. Matrix consistency is defined as follows: a positive $n \times n$ matrix A is consistent if $a_{ij}a_{jk} = a_{ik}$, for $i, j, k = 1, \dots, n$. Although different measurements of inconsistency can be developed, in this paper we use the measurement proposed by Saaty [1,3]. We also use the intrinsic consistency threshold developed by Monsuur [10].

If consistency is unacceptable, it should be improved. Several alternatives, mostly based on various optimization techniques, have been proposed in the literature to help improve consistency, including [9,11–21]. For example, Saaty [11] proposes a method based on perturbation theory to find the most inconsistent judgment in the matrix. This action could be followed by the determination of the range of values to which that judgment can be changed and whereby the inconsistency could be improved – and then asking the judge to consider changing the judgment to a plausible value in that range.

In the next section we develop a linearization technique that provides the closest consistent matrix to a given non-consistent matrix by using an orthogonal projection in a given linear space. Our method provides a closed form for achieving consistency, while methods relying on optimisation, which is non-linear for this problem, are iterative by nature. Section 3 presents an application to a real-world decision-making problem regarding leakage policies in water supply. Finally, the paper closes with conclusions.

2. Achieving consistency through linearization

From now on, $M_{n,m}$ and $M^+_{n,m}$ will denote the set of $n \times m$ matrices and the set of $n \times m$ positive matrices, respectively. It will be assumed that the elements of $\mathbb{R}^n$ are column vectors. For a given A, the entry (i,j) of A will be denoted by $[A]_{i,j}$. Furthermore, A^T denotes the transposition of the matrix A. The matrix product component-wise (also called the Hadamard product) of A, B is the matrix $A \odot B$ defined by $[A \odot B]_{i,j} = [A]_{i,j}[B]_{i,j}$. The (nonlinear) map J: $M_{n,m} \to M_{n,m}$ given by $[J(A)]_{i,j} = 1/[A]_{i,j}$ will be useful. In particular, notice that $A \in M^+_{n,n}$ is reciprocal if and only if $J(A) = A^T$.

In the following lines we linearize the problem of finding a consistent matrix close to a given positive matrix. The mathematical tool to measure the closeness of two given matrices is the concept of matrix norm (see e.g. [22, section 5.2]). Here we use the Frobenius norm because of its simplicity. Such a norm is defined as

$$\|A\|_F = \left(\sum_{i,j}[A]^2_{i,j}\right)^{1/2} = \left[\text{trace}(A^TA)\right]^{1/2}, \quad A \in M_{n,n}.$$

Furthermore, let us define the following map:

$$L : M^+_{n,n} \to M_{n,n}, \quad [L(X)]_{i,j} = \log([X]_{i,j}).$$

Obviously, this map is bijective (one to one) and satisfies $L(X \odot Y) = L(X) + L(Y)$ for all $X, Y \in M^+_{n,n}$. The following map is the inverse of L:

$$E : M_{n,n} \to M^+_{n,n}, \quad [E(X)]_{i,j} = \exp([X]_{i,j}).$$

This map satisfies $E(X + Y) = E(X) \odot E(Y)$ for all X, $Y \in M_{n,m}$.

We can characterize reciprocal and consistent matrices by using the map L. Observe that in [12] it is proven that a matrix $A \in M^+_{n,n}$ is consistent if, and only if, there exists $\mathbf{w} = (w_1, \dots, w_n)^T \in M^+_{n,1}$ such that $[A]_{i,j} = w_i/w_j$ for all $1 \leqslant i, j \leqslant n$.

Theorem 2.1. *Let $A \in M^+_{n,n}$.*

(i) *A is reciprocal if, and only if, $L(A)$ is skew-Hermitian.*
(ii) *A is consistent if, and only if, there exists $\mathbf{v} = (v_1, \dots, v_n)^T \in \mathbb{R}^n$ such that $[L(A)]_{i,j} = v_i - v_j$ for all $i, j \in \{1,2,\dots,n\}$.*

Let $\mathcal{A}_n$ and $\mathcal{S}_n$ denote the subsets of $M_{n,n}$ composed of skew-Hermitian matrices and Hermitian matrices, respectively. In the following result we give the key properties of the set

$$\mathcal{L}_n = \left\{L(A) : A \in M^+_{n,n},\ A \text{ is consistent}\right\}.$$

To this end, it will be useful to define the following map:

$$\phi : \mathbb{R}^n \to M_{n,n}, \quad [\phi(\mathbf{x})]_{i,j} = x_i - x_j, \quad \mathbf{x} = (x_1, \dots, x_n)^T. \tag{1}$$

Theorem 2.2. *The set $\mathcal{L}_n$ is a linear subspace of $M_{n,n}$ whose dimension equals $n - 1$.*

Proof. The map defined in (1) is obviously linear and $\text{Im}\phi = \mathcal{L}_n$. Also, it should be evident that $\ker \phi = \text{span}\{(1,\dots,1)^T\}$. Thus, $\dim\mathcal{L}_n = \dim\text{Im}\phi = \dim\mathbb{R}^n - \dim\ker\phi = n - 1$. □