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Balancing consistency and expert judgment in AHP

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ABSTRACT

The various mechanisms that represent the know-how of decision-makers are exposed to a common weakness, namely, a lack of consistency. To overcome this weakness within AHP (analytic hierarchy process), we propose a framework that enables balancing consistency and expert judgment. We specifically focus on a linearization process for streamlining the trade-off between expert reliability and synthetic consistency. An algorithm is developed that can be readily integrated in a suitable DSS (decision support system). This algorithm follows an iterative feedback process that achieves an acceptable level of consistency while complying to some degree with expert preferences. Finally, an application of the framework to a water management decision-making problem is presented.

1. Introduction

One of the best established and most modern models of decision-making is AHP (analytic hierarchy process) [1–3]. In AHP, the input format for decision-makers to express their preferences derives from pair-wise comparisons among various elements. Comparisons can be determined by using, for instance [4], a scale of integers 1–9 to represent opinions ranging from 'equal importance' to 'extreme importance' [5] (intermediate decimal values are sometimes useful). Homogeneous and reciprocal judgment yields an $n \times n$ matrix A with $a_{ii} = 1$ and $a_{ij} = 1/a_{ji},\ i, j = 1, \ldots, n$. This last property is called *reciprocity* and A is said to be a *reciprocal matrix*. The aim is to assign to each of n elements, E_i, priority values $w_i,\ i = 1, \ldots, n$, that reflect the emitted judgments. If judgments are consistent, the relations between the judgments a_{ij} and the values w_i turn out to be $a_{ij} = w_i/w_j,\ i, j = 1, \ldots, n$, and it is said that A is a *consistent matrix*. This is equivalent to $a_{ij}a_{jk} = a_{ik}$ for $i, j,\ k = 1, \ldots, n$ [6]. As stated by [7,2], the leading eigenvalue and the principal (Perron) eigenvector of a comparison matrix provides information to deal with complex decisions, the normalized Perron eigenvector giving the sought priority vector. In the general case, however, A is not consistent. The hypothesis that the estimates of these values are small perturbations of the 'right' values guarantees a small perturbation of the eigenvalues (see, e.g., [8]). Now, the problem to solve is the eigenvalue problem $A\mathbf{w} = \lambda_{\max}\mathbf{w}$, where $\lambda_{\max}$ is the unique largest eigenvalue of A that gives the Perron eigenvector as an estimate of the *priority vector*.

As a measurement of inconsistency, Saaty [5] proposed using the consistency index $CI = (\lambda_{\max} - n)/(n - 1)$ and the consistency ratio $CR = CI/RI$, where RI is the so-called average consistency index [5]. If $CR < 0.1$, the estimate is accepted; otherwise, a new comparison matrix is solicited until $CR < 0.1$. To overcome inconsistency in AHP while still taking into account expert know-how, the authors propose a model to balance the latter with the former. Our model incorporates an extended version of the linearization procedure described in [9], and integrates it along with AHP to produce optimal comparison matrices.

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2. Linearization process and an extension to judgment modification

We first introduce some notation and mathematical tools [9]. Hereinafter, $M_{n,m}$ will denote the set of $n \times m$ real matrices. If $A \in M_{n,m}$, $[A]_{i,j}$ denotes the i, j entry of A. $M^+_{n,m} \subset M_{n,m}$ is the subset of matrices with positive entries. We assume that vectors of $\mathbb{R}^n$ are columns, and denote $\mathbf{1}_n = [1 \cdots 1]^{\mathrm{T}} \in \mathbb{R}^n$. Let us recall that the Hadamard product, $\odot$, in $M_{n,m}$ is defined as the component-wise product. The following two mappings are one inverse of the other:

$$L : M^+_{n,m} \to M_{n,m}, \qquad [L(A)]_{i,j} = \log([A]_{i,j});$$
$$E : M_{n,m} \to M^+_{n,m}, \qquad [E(A)]_{i,j} = \mathrm{e}^{[A]_{i,j}}.$$

Clearly, $L(X \odot Y) = L(X) + L(Y)$, and $E(X) + E(Y) = E(X \odot Y)$ for all $X, Y \in M^+_{n,m}$. Because of its simplicity, we use the Frobenius norm, $\|A\|_F = [\mathrm{tr}(A^{\mathrm{T}}A)]^{1/2}$, $\mathrm{tr}(X)$ and X^{T} being the trace and the transpose of matrix X, respectively. Also, in $M^+_{n,m}$ we define the distance d given by $\mathrm{d}(A, B) = \|L(A) - L(B)\|_F$. Finally, we define $\phi_n : \mathbb{R}^n \to M_{n,n}$ given by $[\phi_n(\mathbf{x})]_{i,j} = x_i - x_j$ and $\mathscr{L}_n = \{L(A) : A \in M^+_{n,n}, A \text{ is consistent}\}$.

Theorem 1 ([9]). *$\mathscr{L}_n = \mathrm{Im}\,\phi_n$ is a linear subspace of $M_{n,n}$ of dimension $n - 1$.*

We will now use orthogonal projections, to solve approximation problems. Let $p_n : M_{n,n} \to \mathscr{L}_n$ be such a projection, and let us assume that $\mathbb{R}^n$ is endowed with the standard inner product, inducing the Euclidean norm, and $M_{n,n}$ is endowed with the following inner product: $\langle A, B\rangle = \mathrm{tr}(A^{\mathrm{T}}B)$.

Theorem 2 ([9]). *Let $\{\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}\}$ be an orthogonal basis of the orthogonal complement to $\mathrm{span}\{\mathbf{1}_n\}$. Then $\{\phi_n(\mathbf{y}_1), \ldots, \phi_n(\mathbf{y}_{n-1})\}$ is an orthogonal basis of $\mathscr{L}_n$ and $\|\phi_n(\mathbf{y}_i)\|_F^2 = 2n\|\mathbf{y}_i\|_2^2$ for all $i = 1, \ldots, n-1$.*

Hence, the orthogonal projection of $L(A)$ onto $\mathscr{L}_n$ is given by a Fourier expansion [10].

Theorem 3 ([9]). *Let A and $\{\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}\}$ an orthogonal basis of the orthogonal complement to $\mathrm{span}\{\mathbf{1}_n\}$. The orthogonal projection of $L(A)$ onto $\mathscr{L}_n$ is the matrix*

$$\frac{1}{2n}\sum_{i=1}^{n-1}\frac{\mathrm{tr}(L(A)^{\mathrm{T}}\phi_n(\mathbf{y}_i))}{\|\mathbf{y}_i\|_F^2}\phi_n(\mathbf{y}_i).$$

Remark 4. Observe that $\phi_n(\mathbf{v}) = \mathbf{v}\mathbf{1}_n^{\mathrm{T}} - \mathbf{1}_n\mathbf{v}^{\mathrm{T}}$ for any $\mathbf{v} \in \mathbb{R}^n$.

We develop now some results that enable easy calculation of the new consistent comparison matrix if one or more judgments are modified. As a corollary, we give a fast algorithm to find the closest consistent matrix to a given reciprocal matrix.

2.1. Consistency retrieval after modifying one pair-wise comparison

Let us suppose that a reciprocal matrix A is obtained from some expert judgment and the consistent matrix $Y_A = E[p_n(L(A))]$ closest to A is calculated. If the judgment comparing criteria r and s is changed (where $r \neq s$ and $1 \le r, s \le n$), we obtain another reciprocal matrix B. In other words, $[B]_{r,s} = \alpha[A]_{r,s}$ and $[B]_{s,r} = \alpha^{-1}[A]_{s,r}$ for some $\alpha > 0$ and $[B]_{i,j} = [A]_{i,j}$ in the remaining entries.

The problem we address is how to find the consistent matrix $Y_B = E[p_n(L(B))]$ closest to B by performing fewer operations than by means of Theorem 3.

The relationship between matrices A and B is

$$L(B) = L(A) + \log\alpha(\mathbf{e}_r\mathbf{e}_s^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_r^{\mathrm{T}}). \tag{1}$$

Since the orthogonal projection p_n is linear,

$$p_n(L(B)) = p_n(L(A)) + \log\alpha \cdot p_n(\mathbf{e}_r\mathbf{e}_s^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_r^{\mathrm{T}}). \tag{2}$$

By Theorem 3 we have

$$p_n(\mathbf{e}_r\mathbf{e}_s^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_r^{\mathrm{T}}) = \frac{1}{2n}\sum_{i=1}^{n-1}\frac{\mathrm{tr}\left((\mathbf{e}_r\mathbf{e}_s^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_r^{\mathrm{T}})^{\mathrm{T}}\phi_n(\mathbf{y}_i)\right)}{\|\mathbf{y}_i\|_2^2}\phi_n(\mathbf{y}_i). \tag{3}$$

Let us simplify $\mathrm{tr}\left((\mathbf{e}_r\mathbf{e}_s^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_r^{\mathrm{T}})^{\mathrm{T}}\phi_n(\mathbf{y}_i)\right)$. By using Remark 4, we obtain

$$(\mathbf{e}_r\mathbf{e}_s^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_r^{\mathrm{T}})^{\mathrm{T}}\phi_n(\mathbf{y}_i) = (\mathbf{e}_r^{\mathrm{T}}\mathbf{y}_i)\mathbf{e}_s\mathbf{1}_n^{\mathrm{T}} - \mathbf{e}_s\mathbf{y}_i^{\mathrm{T}} - (\mathbf{e}_s^{\mathrm{T}}\mathbf{y}_i)\mathbf{e}_r\mathbf{1}_n^{\mathrm{T}} + \mathbf{e}_r\mathbf{y}_i^{\mathrm{T}}.$$