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A linear programming approximation to the eigenvector method in the analytic hierarchy process ☆

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ABSTRACT

Eigenvector method (EM) is a well-known approach to deriving priorities from pairwise comparison matrices in the analytic hierarchy process (AHP), which requires the solution of a set of nonlinear eigenvalue equations. This paper proposes an approximate solution approach to the EM to facilitate its computation. We refer to the approach as a linear programming approximation to the EM, or LPAEM for short. As the name implies, the LPAEM simplifies the nonlinear eigenvalue equations as a linear programming for solution. It produces true weights for perfectly consistent pairwise comparison matrices. Numerical examples are examined to show the validity and effectiveness of the proposed LPAEM and its significant advantages over a recently developed linear programming method entitled LP-GW-AHP in rank preservation.

1. Introduction

Analytic hierarchy process (AHP) proposed by Saaty [9,10] is a very useful decision-making tool that breaks down complex decision-making problems into hierarchies and conducts decision analysis through pairwise comparison matrices. One of the key steps of applying the AHP for decision analysis is to derive priorities from pairwise comparison matrices. In spite of the fact that a large number of approaches have been proposed such as the eigenvector method (EM) [9,10], the logarithmic least squares method (LLSM) [12,17], the weighted least-squares method (WLSM) [1], the DEAHP approach [6,8], the data envelopment analysis (DEA) method [15,16], the correlation coefficient maximization approach (CCMA) [18], the analytic hierarchy prioritization process (AHPP) [19], the evolutionary computing (EC) approach [20] and the like, the EM is still the most popular approach to weight derivation in the AHP due to its good property of rank preservation.

The EM produces eigenvector weights from pairwise comparison matrices, but requires the solution of a set of nonlinear eigenvalue equations. Since nonlinear equations are not as easy to solve as linear models, the motivation of this paper is thus to develop a linear approximate model for the EM to facilitate its computation. The linear approximate model is a linear programming and is referred to as the linear programming approximation to the EM, or LPAEM for short. The LPAEM turns out to be as good as the EM in rank preservation and to produce extremely close priorities to eigenvector weights, but is easier to solve and more convenient to use. The originality of the paper lies in that the proposed LPAEM offers a simple and new way of deriving priorities from pairwise comparison matrices and makes the application of the AHP easier than before.

The rest of the paper is organized as follows. Section 2 proposes the linear approximate model. Section 3 compares it with recently developed DEA/AR and LP-GW-AHP models. Numerical examples are examined in Section 4 to show the validity and

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effectiveness of the proposed linear approximate model and its significant advantages over the LP-GW-AHP model in rank preservation. Section 5 concludes the paper.

2. LPAEM

Let $A = (a_{ij})_{n\times n}$ be a pairwise comparison matrix with $a_{ii} = 1$ and $a_{ji} = 1/a_{ij}$ for $j \neq i$ and $W = (w_1,\ldots,w_n)^T$ be a priority vector. The eigenvector method produces eigenvector weights by solving the following eigenvalue equations:

$$\sum_{j=1}^{n} a_{ij}w_j = \lambda_{max}w_i, \quad i = 1,\ldots,n, \tag{1}$$

where $\lambda_{\max}$ is the maximal eigenvalue of the pairwise comparison matrix A and $w_i(i = 1,\ldots,n)$ satisfy $w_i > 0$ for $i = 1,\ldots,n$ and $\sum_{i=1}^{n} w_i = 1$. To elicit a linear approximation for the EM, we rewrite (1) as

$$\sum_{j=1}^{n} a_{ij}(w_j/\lambda_{max}) = w_i, \quad i = 1,\ldots,n \tag{2}$$

and define n new variables as

$$z_j = w_j/\lambda_{max}, \quad j = 1,\ldots,n. \tag{3}$$

Nonlinear Eq. (2) can then be linearized as

$$\sum_{j=1}^{n} a_{ij}z_j = w_i, \quad i = 1,\ldots,n. \tag{4}$$

It is known from AHP theory that for any pairwise comparison matrices of n order, there always exists $\lambda_{\max} \geqslant n$, which means that the maximal eigenvalue has a lower bound value of n. It is also known from the following lemma that $\lambda_{\max}$ has still an upper bound value.

Lemma 1. *Let $A = (a_{ij})_{n\times n}$ be a nonnegative matrix with nonzero row sums $r_1,\ldots,r_n$ and maximal eigenvalue λ_{max}. Then* [5]

$$\min_i \left(\frac{1}{r_i} \sum_{j=1}^{n} a_{ij}r_j \right) \leqslant \lambda_{max} \leqslant \max_i \left(\frac{1}{r_i} \sum_{j=1}^{n} a_{ij}r_j \right). \tag{5}$$

Since the pairwise comparison matrix A and its transpose A^T both have the same maximal eigenvalue, the above inequalities should also hold for the transpose of A. That is

$$\min_i \left(\frac{1}{c_i} \sum_{j=1}^{n} a_{ji}c_j \right) \leqslant \lambda_{max} \leqslant \max_i \left(\frac{1}{c_i} \sum_{j=1}^{n} a_{ji}c_j \right), \tag{6}$$

where $c_1,\ldots,c_n$ are the column sums of A. Let

$$\beta = \min\left\{ \max_i \left(\frac{1}{r_i} \sum_{j=1}^{n} a_{ij}r_j \right), \max_i \left(\frac{1}{c_i} \sum_{j=1}^{n} a_{ji}c_j \right) \right\}. \tag{7}$$

It is easy to see from (5) and (6) that $\lambda_{\max} \leqslant \beta$. Therefore, β can be viewed as the upper bound value of $\lambda_{\max}$. Accordingly, we get $n \leqslant \lambda_{\max} \leqslant \beta$. From (3), it can be further derived that

$$w_j/\beta \leqslant z_j \leqslant w_j/n, \quad j = 1,\ldots,n, \tag{8}$$

which forms an assurance region (AR) for decision variables.

After summing up both sides of (3) from $j = 1$ to n, we get

$$\sum_{j=1}^{n} z_j = 1/\lambda_{max}. \tag{9}$$

It is evident that maximizing the eigenvalue $\lambda_{\max}$ is equivalent to minimizing the sum of $z_j(j = 1,\ldots,n)$. So, the linear approximate model to the EM can finally be formulated as the linear programming below:

$$\text{Minimize } J = \sum_{i=1}^{n} z_i \tag{10}$$

$$\text{Subject to} \quad \begin{cases} \sum_{j=1}^{n} a_{ij}z_j = w_i, \quad i = 1,\ldots,n, \\ z_i - \frac{1}{n}w_i \leqslant 0, \quad i = 1,\ldots,n, \\ z_i - \frac{1}{\beta}w_i \geqslant 0, \quad i = 1,\ldots,n, \\ \sum_{i=1}^{n} w_i = 1, \\ w_i, z_i \geqslant 0, \quad i = 1,\ldots,n, \end{cases}$$