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Mathematical properties of EOQ models with special cost structure

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ABSTRACT

An existence-uniqueness theorem is proved about a minimum cost order for a class of inventory models whose holding costs grow according to a stock level power law. The outcomes of Mingari Scarpello and Ritelli (2008) [1] are then extended to different environments: i.e., when the holding costs change during time generalizing a model available in Weiss (1982) [11], or with invariable holding costs but adopting a backordering strategy. Application cases are provided assuming several functional behaviors of demand versus the stock level.

1. Introduction

In previous papers by two of us [1,2] the economic order quantity (EOQ) determination has been generalized starting from the Giri and Chaudhuri approach, see [3], that the demand of a consumer good depends on the available quantity. In order to see what this means exactly, it will be recalled that EOQ is a set of mathematical models defining the optimal quantity of a single item (or good) which shall be ordered for minimizing the total costs: ordering and inventory holding. These models have been in existence long before the computer, their origin going back in time to Harris [4] even though Wilson [5] is credited for his early in-depth analysis on the subject. Basic underlying assumptions:

1. the monthly (annual or, generally: relevant to unit time) demand for the goods is known and deterministic;
2. no lead time (between order and arrivals) is taken into account;
3. the receipt of the order occurs in a single instant and immediately after ordering it;
4. quantity discounts are not calculated as part of the model;
5. the ordering cost A is a constant.

Several extensions can be made to the basic EOQ model: for instance the deterministic demand of goods can change with the instantaneous stock level or with time; the model can include backordering costs and multiple items. Should they undergo deterioration, the perishability can be assumed: constant or variable with the stock level. Finally, the above determinism could be released, leading to a probabilistic insight, but we will keep out of it. Recently several papers have been issued in the field of EOQ under stock-dependent demand and good perishability. In such a way [6] studies the case of perishability with price and stock-dependent demand [7] operates too with nonlinear holding cost of perishables [8] adds inflation and time value of money, while [9] introduces the problem that EOQ car requires an additional storage facility.

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2. Our contribution

Our contribution consists of building, in the frame of EOQ theory, a cost model where arbitrary functions can be employed in order to describe the consumption dynamics and the goods perishability with backordering too. In the first time costs are described, see Eq. (3.3), without backordering, while Eq. (4.2) Lemma 4.1 models this effect too, with the perishability not depending on the stock level. Sufficient conditions are then established on the consumption dynamics assumed as autonomous, namely depending on the store level only. Such assumptions affect consequently the blowdown process so that they are capable of assuring existence and uniqueness to the batch minimizing the cost, i.e. the EOQ. Such conditions are fulfilled by several models of literature, starting from the Wilson one, [5], including its next improvements and/or evolutions, [3,10,11]. Our extension works and generalizes all the EOQ models where the dynamic of the demand is autonomous i.e. is a function of the inventory's level only.

The approach followed hereinafter is of "geometrical" nature in the sense that quadrature closed form relationships are obtained for the reordering time, the global cost function and the minimum cost, namely the optimality condition in such a way that no previous approximation is inserted, very unlike [3] where *at the beginning* a linearization is done by a truncated series expansion or in [12], where the cost function, Eq. (3) therein is approximated. In our treatment numerical approximations arise *only at the end*, in order to evaluate the economic order quantity Q^* as a solution to a nonlinear, possibly transcendental, equation. We build sound foundations to all the subjects obtaining, in a rather general frame, some sufficient conditions ensuring that the inventory cost function attains a minimum and its uniqueness, namely the EOQ-problem well-posedness. The contribution newness consists of the application of the method of [1] to new meaningful cases, namely the time-dependent holding costs, according to [10,11], or when the store manager follows a backordering strategy.

Backordering has also been recently tackled by several authors, but always under a constant rate of store level change, [13–16] who propose to detect the optimal batch backordering levels without calculus, but founded upon classic inequalities such that they are between the arithmetic and geometric means powered by the methods in [17,18]. Anyway in our very general frame where the stock inventory level is ruled by nonlinear dynamics, the classic approach through the infinitesimal calculus is by no means compulsory.

3. Variable holding costs

The time-depending holding costs were introduced in order to take into account the higher money effort for keeping fresh some perishable goods. Let the stored goods blow down according to law:

$$\begin{cases} \dot{q}(t) = -f(q(t)) \\ q(0) = Q > 0 \end{cases} \tag{3.1}$$

where the function $f : [0, \infty[\to \mathbb{R}$ is assumed positive, so that the solution to (3.1) fulfills $q(t) \leqslant Q$ for each $t \geqslant 0$. The autonomous structure of (3.1) allows a closed form solution: defining

$$F(q) := \int_q^Q \frac{1}{f(u)} \mathrm{d}u = t \tag{3.2}$$

then, inverting $F(q)$ we find that $q(t) = F^{-1}(t)$ solves (3.1).

We call *reordering time* generated by the batch Q the real positive value $T(Q)$ solution of $q(t) = 0$ where $q(t)$ solves (3.1):

$$T(Q) = F(0) = \int_0^Q \frac{1}{f(u)} \mathrm{d}u.$$

If $A > 0$ is the delivery cost, $\hat{h}(t) > 0$ models the holding cost at time t as a *continuous* function, so that $\hat{h}(0) > 0$, if $\hat{k}(q)$ denotes a continuous and positive function of q so that $\hat{k}(q) \to \infty$ for $q \to \infty$ and that $\hat{k}(0) = 0$, then the total cost for reordering an amount $Q > 0$ of goods is:

$$C(Q) = \frac{A}{T(Q)} + \frac{h}{T(Q)} \int_0^{T(Q)} \hat{h}(t)\hat{k}(q(t))\mathrm{d}t. \tag{3.3}$$

The Wilson originary treatment, [5] follows putting $f(u) = \delta, \hat{h}(t) = h, \hat{k}(q) = q$. Notice that several literature models: [3,10,11] are all particular cases of what is above, being there $f(u) = au + bh^{\beta}, \hat{h}(t) = ht^{\alpha}$. In [1] is treated as the case for whichever $f(u)$.

Theorem 3.1. *Suppose that function f in (3.1) is such that*

$$\lim_{v \to \infty} \int_0^v \frac{\mathrm{d}u}{f(u)} = \infty. \tag{3.4}$$

Moreover we assume that if $f(0) = 0$, the integrability in $u = 0$ of both functions: