Linear Algebra and its Applications 437 (2012) 294–306

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

An open problem on inverse matrices from industrial organization, and a partial solution

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ARTICLE INFO

Article history:
Received 5 November 2011
Accepted 22 February 2012
Available online 17 March 2012

Submitted by R. Brualdi

AMS classification:
Primary: 15A29, 15B99
Secondary: 91A10, 91B24

Keywords:
Arbitrary coalition structure
Inverse matrices
Merger
Multiproduct oligopoly
Strategic equilibrium

ABSTRACT

This note formulates a large class of matrices whose inverses form an open research problem. It also provides a partial solution as a starting point to tackle the problem in future studies.

1. Introduction

The formation and stability of a coalition structure (or a partition) is an important and unsettled issue in both sciences and social sciences, such as artificial intelligence (Monderer and Tennenholtz [5]) and game theory (Shubik [8]). An obstacle in resolving this issue is that one needs to obtain the analytical expressions of equilibrium payoffs for an arbitrary coalition structure, which often requires one to invert matrices whose inverses are unknown. This note derives a large class of such matrices from industrial organization whose inverses yield the strategic equilibria (or Nash equilibria [6]) in most linear oligopoly models. Such inverses form an open problem, which includes *the analytical expression of the inverse for a general symmetric matrix*.

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http://dx.doi.org/10.1016/j.laa.2012.02.019

As a starting point to find a complete solution in future studies, the paper provides a partial solution by inverting a non-trivial subset of the derived matrices. This partial solution, previously unavailable in the literature, will be useful to other scholars in their future studies.[1] The rest of the note is organized as follows: Section 2 defines and Section 3 derives the open problem, Sections 4 and 5 provide a partial solution and an application, Section 6 concludes, and the appendix provides proofs.

2. Description of the problem

Given a partition (or a coalition structure or a set of multi-product firms) $\Delta = \{S_1, S_2, \dots, S_h\}$ of $N = \{1, \dots, n\}$, let $n_i = |S_i|$ denote the cardinality of each coalition S_i (or the number of products in S_i, $1 \leqslant n_i \leqslant n$, $\sum_{i=1}^{h} n_i = n$), and h_1 the number of its singleton coalitions (i.e., those S_i with $n_i = 1$). Then, our main matrix B has $[n + h(h+1)/2 - h_1]$ constants distributed as below: there are h symmetric $n_i \times n_i$ submatrices B_{ii} on the main diagonal whose diagonal entries are a_k ($k \in S_i$) and off-diagonal entries are a constant $-b_i$ ($i = 1, \dots, h$ and $n_i \geqslant 2$), and $h(h-1)$ other submatrices $B_{ij} = B_{ji}^{\top}$ with a dimension of $n_i \times n_j$ and an identical entry of $-c_{ij} = -c_{ji}$ ($j = 1, \dots, h, j \neq i$). Precisely, the matrix B has the following structure:

$$B = B_{n\times n} = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1h} \\ B_{21} & B_{22} & \cdots & B_{2h} \\ \vdots & \vdots & \ddots & \vdots \\ B_{h1} & B_{h2} & \cdots & B_{hh} \end{pmatrix}, \tag{1}$$

whose block submatrices B_{ii} and B_{ij} ($i, j = 1, \dots, h, j \neq i$) are defined as

$$B_{ii} = \begin{pmatrix} a_{\hat{i}+1} & -b_i & \cdots & -b_i \\ -b_i & a_{\hat{i}+2} & \cdots & -b_i \\ \vdots & \vdots & \ddots & \vdots \\ -b_i & -b_i & \cdots & a_{\hat{i}+n_i} \end{pmatrix}_{n_i\times n_i} \quad \text{and} \quad B_{ij} = \begin{pmatrix} -c_{ij} & -c_{ij} & \cdots & -c_{ij} \\ -c_{ij} & -c_{ij} & \cdots & -c_{ij} \\ \vdots & \vdots & \ddots & \vdots \\ -c_{ij} & -c_{ij} & \cdots & -c_{ij} \end{pmatrix}_{n_i\times n_j},$$

where $\hat{i} = \sum_{k=1}^{i-1} n_k$ is the number of rows (or columns) preceding B_{ii}; $a_k > 0$ ($k = 1, \dots, n$), $b_i > 0$ ($i = 1, \dots, h$ and $n_i \geqslant 2$), and $c_{ij} = c_{ji} > 0$ ($i, j = 1, \dots, h$ and $j \neq i$) are $[n + h(h+1)/2 - h_1]$ constants whose economic interpretations are provided in (5) in the next section.[2]

As an example, the matrix for $\Delta = \{[1, 2], [3, 4], [5]\}$ with $S_1 = \{1, 2\}, S_2 = \{3, 4\}$, and $S_3 = \{5\}$ (i.e., $n = 5, h = 3, n_1 = 2, n_2 = 2, n_3 = 1$, and $h_1 = 1$) has $[n + h(h+1)/2 - h_1] = [5 + 3(3 + 1)/2 - 1] = 10$ constants and is given by

$$B_{5\times 5} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} = \begin{pmatrix} a_1 & -b_1 & -c_{12} & -c_{12} & -c_{13} \\ -b_1 & a_2 & -c_{12} & -c_{12} & -c_{13} \\ -c_{12} & -c_{12} & a_3 & -b_2 & -c_{23} \\ -c_{12} & -c_{12} & -b_2 & a_4 & -c_{23} \\ -c_{13} & -c_{13} & -c_{23} & -c_{23} & a_5 \end{pmatrix}, \tag{2}$$

[1] A special case of this partial solution has already been used in several recent studies such as Ishibashi [4] and Wang and Zhao [9].

[2] Note that there will be no b_i in B_{ii} if $n_i = 1$ or S_i is a singleton. The example in (2) has no b_3 in B_{33}, because of $n_3 = 1$.