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# On extremal solutions of inclusion problems with applications to game theory

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**Abstract**

A recursion principle, generalized iteration methods and the axiom of choice are applied to prove the existence of extremal fixed points of set-valued mappings in posets, extremal solutions of an inclusion problem, and extremal Nash equilibria for a normal-form game.


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*Keywords:* Poset; Set-valued mapping; Fixed point; Inclusion problem; Solution; Maximal; Minimal; Sup-center; Inf-center; Normal-form game; Nash equilibrium; Recursion principle; Generalized iteration methods; Order compact; Chain complete; Upper semi-closed

## 1. Introduction

Let $P$ be a non-empty partially ordered set (poset). As an introductory result we show that a set-valued mapping $\mathcal{F}$ from $P$ to the set $2^P \setminus \emptyset$ of non-empty subsets of $P$ has minimal and maximal fixed points, that is, the set Fix $\mathcal{F} = \{x \in P \mid x \in \mathcal{F}(x)\}$ has minimal and maximal elements, if the following conditions hold.

(c1) $\sup\{c, y\} \in P$ for some $c \in P$ and for every $y \in P$.
(c2) If $x \le y$ in $P$, then for every $z \in \mathcal{F}(x)$ there exists a $w \in \mathcal{F}(y)$ such that $z \le w$, and for every $w \in \mathcal{F}(y)$ there exists a $z \in \mathcal{F}(x)$ such that $z \le w$.
(c3) Strictly monotone sequences of $\mathcal{F}[P] = \bigcup\{\mathcal{F}(x) : x \in P\}$ are finite.

As for the proof, denote $x_0 = c$, and choose $y_0$ from $\mathcal{F}(x_0)$. If $y_0 \not\le x_0$, then $x_0 < x_1 := \sup\{c, y_0\}$. Apply then condition (c2) to choose $y_1$ from $\mathcal{F}(x_1)$ such that $y_0 \le y_1$. If $y_0 = y_1$, then stop. Otherwise, $y_0 < y_1$, whence $x_1 = \sup\{c, y_0\} \le x_2 := \sup\{c, y_1\}$, and apply again condition (c2) to choose $y_2$ from $\mathcal{F}(x_2)$ such that $y_1 \le y_2$. Continuing in the similar way, condition (c3) ensures that after a finite number of choices we get the situation, where $y_{n-1} = y_n \in \mathcal{F}(x_n)$. In view of the above construction we then have $x_n := \sup\{c, y_{n-1}\} = \sup\{c, y_n\}$.

Denoting $z_0 := x_n$ and $w_0 := y_n$ then $w_0 \in \mathcal{F}(z_0)$ and $w_0 \le \sup\{c, w_0\} = z_0$. If $w_0 = z_0$, then $z_0$ is a fixed point of $\mathcal{F}$. Otherwise, denoting $z_1 := w_0$, we have $z_1 < z_0$. In view of condition (c2) there exists a $w_1 \in \mathcal{F}(z_1)$ such that

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$w_1 \leq w_0$. If equality holds, then $z_1 = w_0 = w_1 \in \mathcal{F}(z_1)$, so that $z_1$ is a fixed point of $\mathcal{F}$. Otherwise, $w_1 < w_0$, denote $z_2 := w_1$, and choose by (c2) such a $w_2 \in \mathcal{F}(z_2)$ that $w_2 \leq w_1$, and so on. Condition (c3) implies that a finite number of steps yield the situation $z_m := w_{m-1} = w_m \in \mathcal{F}(z_m)$. Thus $z_m$ belongs to Fix $\mathcal{F}$. Being a subset of $\mathcal{F}[P]$, strictly monotone sequences of Fix $\mathcal{F}$ are finite by condition (c3). This property implies in turn that Fix $\mathcal{F}$ has minimal and maximal elements.

The above described result will be generalized in Section 3. For instance, we show that $\mathcal{F}$ has minimal and maximal fixed points when the above conditions (c1) and (c2) hold and condition (c3) is replaced by order compactness of the values of $\mathcal{F}$ and relative chain completeness of its range $\mathcal{F}[P]$. The obtained results are then used in Section 4 to generalize existence results derived in [5–7] for inclusion problem $Lu \in \mathcal{N}u$, where $L$ is a single-valued mapping from a poset $V$ to $P$, and $\mathcal{N}$ is a set-valued mapping from $V$ to $2^P \setminus \emptyset$. Finally, in Section 5 results of Section 3 are applied to study the existence of extremal Nash equilibria for a normal-form game. Existence proofs require several consecutive applications of a recursion principle and generalized iteration methods introduced in [4,8] and presented in Section 2.

## 2. Preliminaries

In this section $P = (P, \leq)$ is a non-empty poset. When $z \in P$, denote

$$[z) = \{x \in P : z \leq x\} \quad \text{and} \quad (z] = \{x \in P : x \leq z\}.$$

We say that $P$, equipped with a topology is an *ordered topological space* if the order intervals $[z)$ and $(z]$ are closed for each $z \in P$. If the topology of $P$ is induced by a metric, we say that $P$ is an *ordered metric space*.

We say that a subset $W$ of $P$ is *well-ordered* if every non-empty subset of $W$ has the least element. A well-ordered set is a chain, i.e. totally ordered.

A basis to our considerations is the following recursion principle (cf. [8, Lemma 1.1.1]).

**Lemma 2.1.** *Given a subset $\mathcal{D}$ of $2^P$ with $\emptyset \in \mathcal{D}$ and a mapping $f : \mathcal{D} \to P$, there is a unique well-ordered chain $C$ in $P$ such that*

$$x \in C \text{ if and only if } x = f(C^{<x}), \quad \text{where } C^{<x} = \{y \in C \mid y < x\}. \tag{2.1}$$

*If $C \in \mathcal{D}$, then $f(C)$ is not a strict upper bound of $C$.*

*Hint to the proof.* The well-ordered chains $W$ of $P$ whose elements satisfy $x = f(W^{<x})$ are nested, and $C$ is their union. □

As an application of Lemma 2.1 we get the following result (cf. [4, Lemma 2]).

**Lemma 2.2.** *Given $G : P \to P$ and $c \in P$ there exists a unique well-ordered chain $C = C(G)$ in $P$, called a w.o. chain of $cG$-iterations, satisfying*

$$x \in C \text{ if and only if } x = \sup\{c, G[C^{<x}]\}. \tag{2.2}$$

**Proof.** Denoting $\mathcal{D} = \{W \subset P : W \text{ is well-ordered and } \sup\{c, G[W]\} \text{ exists}\}$, and defining $f(W) = \sup\{c, G[W]\}$, $W \in \mathcal{D}$, we obtain a mapping $f : \mathcal{D} \to P$, and (2.2) is reduced to (2.1). Thus, by Lemma 2.1 there is a unique well-ordered chain $C$ in $P$ with (2.2). □

A subset $W$ of a chain $C$ is called an *initial segment* of $C$ if $x \in W$ and $y < x$ imply $y \in W$. The following application of Lemma 2.1 is also used in the sequel.

**Lemma 2.3.** *Let $\mathcal{F} : P \to 2^P \setminus \emptyset$ and $c \in P$ be given. Denote by $\mathcal{G}$ the set of all selections $G$ from $\mathcal{F}$, i.e.,*

$$\mathcal{G} := \{G : P \to P \mid G(x) \in \mathcal{F}(x) \text{ for all } x \in P\}. \tag{2.3}$$

*For every $G : P \to P$ denote by $C_G$ the longest such an initial segment of the w.o. chain $C(G)$ of $cG$-iterations that the restriction $G|C_G$ of $G$ to $C_G$ is increasing. Define a partial ordering $\prec$ on $\mathcal{G}$ as follows.*

(O) *$F \prec G$ if and only if $C_F$ is a proper initial segment of $C_G$ and $G|C_F = F|C_F$.*

*Then $(\mathcal{G}, \preceq)$ has a maximal element.*