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Pricing fuzzy vulnerable options and risk management

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ABSTRACT

The assumption of unrealistic "identical rationality" in classic option pricing theory is released in this article to amend Klein's [Klein, P. (1996). Pricing Black–Scholes options with correlated credit risk. *Journal of Banking Finance*, 1211–1129] vulnerable option pricing formula. Through this formula, default risk and liquidity risk are both well-explained when the investment behaviors and market expectations of the participants are heterogeneous. The numerical results show that when the investing decisions of each market participant come from their individual rationality and use their own subjective price to trade, the option price becomes a boundary. The upper boundary becomes an absolutely safe line and the lower boundary becomes an absolutely unsafe line for investors who want to invest in some financial securities with default risk. The proposed model suggests a more realistic pricing mechanism for the issuers and traders who want to value options with default risk.

1. Introduction

The most important thing about deals that are traded in the over-the-counter (OTC) market is the associated credit risk, since there is no exchange or clearing house to ensure that investors will honor their obligations. There are many reasons why people may not do so. First, perhaps due to regret at making the wrong financial decisions. Second, due to not having enough assets to cover the investment loss. Third, perhaps their assets are enough to pay for the claim, but they lack the necessary liquidity. Hence, derivatives traded in the OTC market typically not only face the risk of default, but also the risk of illiquidity, and this is gaining more and more attention from both theorists and practitioners. The definition of market liquidity is that people are able to unwind their large holding positions at reasonable costs. Investors always prefer holding assets which are more liquid, since they can sell them at any time, especially when market prices are high. When an asset is illiquid, it means that even if it has good market price, it perhaps can be sold at the right time.

Since options are not always risk free, the holders are vulnerable to default risk. These kinds of options are called "vulnerable options", which, as first proposed by Johnson and Stulz (1987), is an option whose counterparty may default on the obligations it has in the contract. The vulnerable option pricing formula of Johnson and Stulz (1987) assumes that the option is the only liability of the firm and the default occurs when the value of the option outgrows the value of the assets of the counterparty. Following this, Hull and White (1995) proposed a model to price vulnerable options to allow the counterparty to have other liabilities of equal priority. When default occurs, only a proportion of the original claims are paid to the option holders. Jarrow and Turnbull (1995) provided a new methodology for pricing and hedging derivative securities subject to default risk, and their model can be applied to securities other than vulnerable options. Rich (1996) developed a model to price European options subject to an intertemporal default risk, considering the timing of the default and an uncertain recovery value, and it can also be used to evaluate the current margin requirements made for exchange-trade options. Klein (1996) criticized Johnson and Stulz's (1987) assumption that the option could be treated as the only liability in the option writer's capital structure, improving the earlier model to allow the option writer to have other liabilities of equal priority payment. However, the underlying assets of the vulnerable option, such as firm value or collateral, are usually illiquid, and treating them as liquid assets when pricing vulnerable options is inappropriate. Hence, a model must be developed to accommodate the incompleteness of the real market to price vulnerable options.

Cherubini and Lunga (2001a, 2001b) used fuzzy measure theory to represent liquid risk and establish a fuzzified version of the Merton (1974) model. Their model enables one to account for different values of long and short positions, and liquidity risk is introduced by representing the upper and lower bound of the price of the contingent claim computed as the upper and lower Choquet integral with respect to a sub-additive function. Han and Zheng (2005) noted that even though Cherubini and Lunga (2001a, 2001b) dealt with the fuzzy measure and integral, they did not view this problem from an economic perspective, and did not account for the "non-identical rationality" concept. Hence, they went back to consider the basic assumptions of classical economics in

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constructing financial models, and pointed out that all the mispricing of previous models comes from the unrealistic assumption that all the investors in the markets are "identical rationality" by using a representative agent for the market. Under this assumption, the preference, expectation and investment strategy of all the investors' are homogenous, and all the investors' interest can be maximized through a representative agent. Many empirical studies find that there are systematic errors between the real market and the classical model-based price. Phenomena like violation for put-call parity, volatility smile, volatility term structure and so on, are all evidence of the breakdown of the classical models. Some researchers have proposed different approaches, such as the pure jump model, jump diffusion model, and stochastic volatility model to correct these defects, but they fail because they do not address the unrealistic assumption – all the market participants have "identical rationality". Leland (1980) suggested that if all the market participants are all "identical rationality" and their long and short position are homogenous, there will be no trade in the market. Hence, the existence of both heterogeneity and non-identical rationality among market participants must be faced squarely when we want to establish a realistic economic or a financial model.

This paper follows the idea of Cherubini and Lunga (2001a, 2001b) and employs the method of Han and Zheng (2005) and the model of Klein (1996) to develop a method to price vulnerable options by using the fuzzy measure theory to represent liquidity risk. We focus on the case in which the probability measure used to price contingent claims is not known precisely. This theory enables us to explain for different values of long and short positions. Liquidity risk can be thought of as resulting from the non-identical rationality of all the investors, with their own prices different from others, leading to different market positions (long or short). The use of a specific class of fuzzy measures, known as g_λ-measures, enables us to easily extend Klein's (1996) model to the case of illiquid markets. As the technique is particularly useful in corporate claims evaluation, a fuzzified version of Klein's (1996) model of liquidity risk is presented.

The article is structured as follows. Section 1 presents an overview of research on default risk models and the superiority of the fuzzy measure theory in dealing with the problems of liquidity risk, which has been seldom discussed in previous articles. Section 2 introduces Klein's (1996) model, the fuzzy measure theory, and amends it to account for liquidity risk. Section 3 numerically analyzes the proposed model and value-at-risk in a fuzzy environment. Finally, Section 4 draws conclusions.

2. The model

In this section, we follow Klein (1996) to build up a fuzzified version of vulnerable option pricing model. This model resolves the problems in Johnson and Stulz (1987) and Klein (1996) which treat illiquid or not tradable assets, such as the collateral assets in their articles, as liquid or tradable ones. We adopt the idea of Cherubini and Lunga (2001a, 2001b) and the pricing method of Han and Zheng (2005) to develop a method to price vulnerable options by using the fuzzy measure theory to represent liquidity risk. We focus on the case in which the probability measure used to price contingent claims is not known precisely. This theory enables us to explain investors' different values of long and short positions.

2.1. Klein's (1996) model

Klein's (1996) model can be expressed as follows. Assume that V is the value of the assets of the counterparty, which is large compared to the expected payoff under the option. V is defined to include the current market value of all assets of the counterparty and the marked to market value of all derivatives and other contracts, including the writer's position in the option being valued and any hedging related to it. If the option expires at time T and V_T is less than the critical value of claims, D^*, then default occurs, otherwise not. D is the total amount associated with claims which is larger than D^* due to the possibility of a counterparty continuing in operation even when the total value of assets of the counterparty at time T, V_T, is less than D. If default occurs, the counterparty pays out only the proportion $\frac{(1-\alpha)}{D}V_T$ of the normal claim B. Thus, the expected actual payout at maturity time T, B^*, is defined as follows:

$$B^* = E^*[B|V_T \geqslant D^*] + E^*\left[B\frac{(1-\alpha)V_T}{D}\middle|V_T < D\right]. \tag{1}$$

Let S be the value of the assets underlying the option. Assume that S and V follow geometric Brownian motion with instantaneous drift μ_S and μ_V, and volatility σ_S and σ_V, respectively. We can express the processes of S and V as follows:

$$\frac{dS}{S} = \mu_S\,dt + \sigma_S\,dz^P, \tag{2}$$

$$\frac{dV}{V} = \mu_V\,dt + \sigma_V\,dw^P, \tag{3}$$

and

$$Cov(dz^P, dw^P) = \rho.$$

Changing the measure from P to the risk-neutral measure Q, we have

$$dz^P = dz^Q - \left(\frac{\mu_S - r}{\sigma_S}\right),$$

$$dw^P = dw^Q - \left(\frac{\mu_V - r}{\sigma_V}\right).$$

Applying Ito's lemma, we can find the processes for $\ln S$ and $\ln V$

$$d\ln S = \left(r - \frac{1}{2}\sigma_S^2\right)dt + \sigma_S\,dz^Q, \tag{4}$$

$$d\ln V = \left(r - \frac{1}{2}\sigma_V^2\right)dt + \sigma_V\,dw^Q. \tag{5}$$

From the equation above, it is clear that $\ln S_T$ and $\ln V_T$ have standard bivariate normal distribution, $N_2(\cdot)$

$$N_2\left(\ln S_t + \left(r - \frac{\sigma_S^2}{2}\right)(T-t), \ln V_t + \left(r - \frac{\sigma_V}{2}\right)(T-t),\right.$$
$$\left.\sigma_S\sqrt{T-t}, \sigma_V\sqrt{T-t}, \rho\right). \tag{6}$$

Thus, the value of vulnerable call option C^* can be expressed as follows:

$$C^* = e^{-r(T-t)}E^*[\max(S_T - K, 0)([1|V_T \geqslant D^*] + [(1-\alpha)V_T/D|V_T < D^*])]. \tag{7}$$

After lengthy computation we can obtain

$$C^* = S_t\left(N_2(a_1, a_2, \rho) + e^{(r+\rho\sigma_S\sigma_V)(T-t)}(1-\alpha)\frac{V_t}{D}N_2(c_1, c_2, -\rho)\right)$$
$$- e^{-r(T-t)}K\left(N_2(b_1, b_2, \rho) + e^{r(T-t)}\frac{V_t}{D}N_2(d_1, d_2, -\rho)\right), \tag{8}$$

where the parameters $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ are as follows:

$$a_1 = \frac{\ln(S_t/K) + (r + \sigma_{S_t}^2/2)(T-t)}{\sigma_S\sqrt{T-t}} = b_1 + \sigma_S\sqrt{T-t},$$