Nonlinear Analysis 75 (2012) 1572–1591

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Min–max game theory and nonstandard differential Riccati equations for abstract hyperbolic-like equations[☆]

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ARTICLE INFO

Communicated by Guest Editor (Chennai)

Keywords:
Min-Max game theory
Non-standard Riccati equations
Hyperbolic dynamics

ABSTRACT

We consider the abstract dynamical framework of Lasiecka and Triggiani (2000) [1, Chapter 9], which models a large variety of mixed PDE problems (see specific classes in the Introduction) with boundary or point control, all defined on a smooth, bounded domain $\Omega \subset R^n$, n arbitrary. This means that the input → solution map is bounded on natural function spaces. We then study min–max game theory problem over a finite time horizon. The solution is expressed in terms of a (positive, self-adjoint) time-dependent Riccati operator, solution of a non-standard differential Riccati equation, which expresses the optimal qualities in pointwise feedback form. In concrete PDE problems, both control and deterministic disturbance may be applied on the boundary, or as a Dirac measure at a point. The observation operator has some smoothing properties.

1. Mathematical setting and formulation of the min–max problem. Statement of main results

In this paper we return to the abstract dynamical setting of [1, Chapter 9] ("abstract hyperbolic-like equations"), which model a large variety of mixed PDE problems of hyperbolic-like type with either boundary or point control/disturbance, all defined on a smooth bounded domain $\Omega \subset R^n$. They include: second-order hyperbolic equations and Schrödinger equations with control/disturbance in the Dirichlet B.C. [1, Sections 10.5 and 10.9]; non-symmetric, non-dissipative, first-order hyperbolic systems with boundary control/disturbance [1, Section 10.6]; Euler–Bernoulli and Kirchhoff plate equations with different types of boundary controls/disturbances [1, Sections 10.7; 10.8], even wave equations with Neumann boundary control/disturbance, however, only in one dimension [1, Section 9.8.4, p. 857]; as well as wave and Kirchhoff equations with point control/disturbance [1, Section 9.8], as well as systems of coupled PDe equations (wave and Kirchhoff equations; wave and structurally damped Euler–Bernoulli equations [1, Sections 9.10 and 9.11] arising in noise reduction models, with point control/disturbance). One may add elastic and thermoelastic dynamics with point control disturbance [2–5]. For such abstract dynamics, we then study a min–max game theory problem over a finite time horizon and with a smoothing observation operator. This is the perfect counterpart of the optimal control problem studied in [1, Chapter 9]. In the solution of this min–max problem that we provide here, all the optimal quantities – control, disturbance, state, observed state – are expressed explicitly in terms of the data of the problem via a time-dependent, positive definite Riccati operator, solution of a non-standard differential Riccati equation. For such hyperbolic-like dynamics, the corresponding min–max problem over infinite time interval was studied in [6,7] for the stable, respectively, unstable cases leading to an algebraic non-standard Riccati equation.

☆ Research partially supported by the National Science Foundation under grant DMS-0104305 and the Air Force Office of Scientific Research under grant FA9550-09-1-0459.

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doi:10.1016/j.na.2011.08.025

For min–max problems for parabolic-like (analytic semigroup), we refer to [8, Section 9.8], [9,10].

For min–max problems in systems of coupled PDEs of different type (parabolic versus hyperbolic), we quote structural acoustic chambers [11]; a fluid–structure interaction model [12], as well as the abstract treatment in [13], see also [14].

Dynamical model (*Setting in* [8, Chapter 9], [15,16]). Let U (control), V (disturbance), Y (state) be separable Hilbert spaces. In this paper, we consider the following abstract dynamical system

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t) + Gw(t) & \text{in } [\mathcal{D}(A^*)]' \\ y(0) = y_0 \in Y. \end{cases} \tag{1.1}$$

Here the function $u \in L_2(0, T; U)$ is the control and $w \in L_2(0, T; V)$ is a deterministic disturbance. The dynamics (1.1) is subject to the following assumptions, to be maintained throughout this paper:

(A.1) $A : Y \supset \mathcal{D}(A) \to Y$ is the infinitesimal generator of a strongly continuous (s.c.) semigroup e^{At} on Y. Without loss of generality, as the dynamics (1.1) is studied here over a finite time interval $[0, T]$, $T < \infty$, at the price of replacing A with a suitable translation of A, we may assume that $A^{-1} \in \mathcal{L}(Y)$.

(A.2) B and G are linear operators satisfying: $B \in \mathcal{L}(U; [\mathcal{D}(A^*)]')$ and $G \in \mathcal{L}(V; [\mathcal{D}(A^*)]')$, respectively. Equivalently, $A^{-1}B \in \mathcal{L}(U; Y)$, and $A^{-1}G \in \mathcal{L}(V; Y)$. Here, $[\mathcal{D}(A^*)]'$ is the dual space of the domain $\mathcal{D}(A^*)$, with respect to the pivot space Y. Thus, e^{At} can be extended as a s.c. semigroup on $[\mathcal{D}(A^*)]'$ as well.

(A.3) The observation operator R is bounded:

$$R \in \mathcal{L}(Y; Z), \tag{1.2}$$

where Z (output space) is another Hilbert space.

(A.4) The (closable) operators $B^*e^{A^*t}$ and $G^*e^{A^*t}$ can be extended (from $\mathcal{D}(A^*)$) to satisfy the 'abstract trace regularity' [1, p. 766], [17–21]:

$$B^*e^{A^*t} : \text{continuous } Y \to L_2(0, T; U); \qquad G^*e^{A^*t} : \text{continuous } Y \to L_2(0, T; V), \tag{1.3a}$$

that is,

$$\int_0^T \|B^*e^{A^*t}y\|_U^2 dt \le C_T\|y\|_Y^2; \qquad \int_0^T \|G^*e^{A^*t}y\|_V^2 dt \le C_T\|y\|_Y^2, \quad y \in Y \tag{1.3b}$$

(estimate (1.3b) is first checked for all $y \in \mathcal{D}(A^*)$ and then extended to all of Y).

(A.5) The maps $R^*Re^{At}B$ and $R^*Re^{At}G$ can be extended as follows [1, p. 767], [21]:

$$R^*Re^{At}B : \text{continuous } U \to L_1(0, T; Y); \qquad R^*Re^{At}G : \text{continuous } V \to L_1(0, T; Y), \tag{1.4a}$$

that is,

$$\int_0^T \|R^*Re^{At}Bu\|_Y dt \le C_T\|u\|_U, \quad u \in U; \qquad \int_0^T \|R^*Re^{At}Gw\|_Y dt \le C_T\|w\|_V, \quad w \in V. \tag{1.4b}$$

Min–max game theory problem on $[0, T]$. For a fixed $0 < T < \infty$ and a fixed $\gamma > 0$, we associate with (1.1) the cost functional

$$J(u, w; y_0) = J(u, w, y(u, w); y_0) = \int_0^T \left[\|Ry(t)\|_Z^2 + \|u(t)\|_U^2 - \gamma^2\|w(t)\|_V^2\right] dt, \tag{1.5}$$

where $y(t) = y(t; y_0)$ is the solution of (1.1) due to $u(t)$ and $w(t)$. See below in (1.15a).

The aim of this paper is to study the following min–max game theory problem:

$$\sup_{w \in L_2(0,T;V)} \inf_{u \in L_2(0,T;U)} J(u, w; y_0), \tag{1.6}$$

where the infimum is taken over all $u \in L_2(0, T; U)$, for $w \in L_2(0, T; V)$ fixed, and the supremum is taken over all $w \in L_2(0, T; V)$. This problem, in addition to being of interest by itself, is known to be the state space formulation of the so called H^∞ robust stabilization problem; see [22,23]. In these references, this problem was introduced and stated in terms of the transfer function in the context of finite-dimensional theory [23]. The min–max game theory problem on the infinite time horizon, $[0, \infty]$ and thus leading to a non-standard algebraic Riccati equation, has been studied in [6,7,24,1]. In this paper, we are interested in a finite time setting, $[0, T]$, and the corresponding non-standard differential Riccati equation.

Dual versions of (A.4), (A.5). By duality in (A.4), we obtain the following relations [8, p. 767]. First, on $B^*e^{A^*t}$:

(A.4*) The map L_0 satisfies [18], [1,21, Chapter 7, Theorem 7.2.1]

$$(L_0u)(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau : \text{continuous } L_2(0, T; U) \to C([0, T]; Y), \tag{1.7}$$