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p-Dominance and perfect foresight dynamics

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Abstract

We investigate stability of **p**-dominant equilibria under perfect foresight dynamics.We show that a strict **p**-dominant equilibrium with $\sum_i p_i < 1$ is globally accessible and absorbing in perfect foresight dynamics. We also investigate robustness and extensions of this result. We apply our proof method to games with u-dominant equilibria and unanimity games.

JEL classification: C72; C73

Keywords: Equilibrium selection; Perfect foresight dynamics; **p**-Dominance

1. Introduction

Perfect foresight dynamics (Matsui and Matsuyama, 1995) is a model of rational and forward-looking individuals. Facing inertia of action revisions, agents in large populations take best responses to the discounted time-average of the action distributions from the present to the future.

Under this dynamics, even a strict Nash equilibrium can be upset by a consistent belief that the society will move away from the equilibrium. This feature makes it possible to use perfect foresight dynamics for equilibrium selection. Specifically, Oyama et al. (2006, OTH henceforth) show that the strict monotone-potential maximizer in a strict monotone-potential game is a *globally accessible* and *absorbing* state if the original game or the strict monotone-potential function is supermodular. That is, there exists a path converging to the state, no matter how far the initial state of the society is, and no path can escape from the state once the path is close to the state. For an n-player asymmetric game and $\mathbf{p} = (p_1, \ldots, p_n)$, an action profile a^* is a (strict) **p**-dominant equilibrium of the game if, for every player i, action a_i^* is a (unique) best response to any belief putting probability at least (or more than) p_i that other players take a_{-i}^*. Since (strict) monotone-potential maximization with supermodular monotone-potential functions generalizes (strict) **p**-dominance with $\sum_{i=1}^n p_i < 1$, OTH's result implies that strict **p**-dominant equilibria with $\sum_i p_i < 1$ are globally accessible and absorbing.

We present an alternative proof for global accessibility and absorption of strict **p**-dominant equilibria with $\sum_i p_i < 1$. For example, global accessibility with zero subjective discount rates is shown as follows. Let a^* be a **p**-dominant equilibrium with $\sum_i p_i < 1$ and x be the initial state. Our proof begins with showing the following: there exists $\mathbf{T} = (T_1, \ldots, T_n)$ such that each agent who receives revision opportunity after T_i puts probability at least p_i, under

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the effectively discounted belief, that the opponent in population j will have received revision opportunity after T_j for every $j \neq i$. Given such $\mathbf{T}$, we consider the restricted set $\Phi(x, \mathbf{T})$ of paths originating at x where agents in population i are required to take a_i^* if they receive revision opportunities after T_i. $\Phi(x, \mathbf{T})$ is nonempty, convex and compact. By the construction of $\mathbf{T}$ and the definition of $\mathbf{p}$-dominance, taking a_i^* after T_i is a best response for agents in population i if agents in population j take a_j^* after T_j for every $j \neq i$. Therefore, the best response correspondence from $\Phi(x, \mathbf{T})$ to itself is a nonempty-valued correspondence. By the fixed point theorem, there is a fixed point of the correspondence, which is a perfect foresight path from x to a^*.

Kajii and Morris (1997) consider an alternative method of equilibrium selection. They say that a Nash equilibrium of a complete information game is robust to incomplete information if every incomplete information game with payoffs almost always given by the complete information game has an equilibrium whose observed behavior is close enough to the Nash equilibrium. They show that any $\mathbf{p}$-dominant equilibrium with $\sum_i p_i < 1$ is robust to incomplete information. Their proof goes as follows. Let Ω be the countable set of states, P be a common prior on Ω, and $\mathcal{Q}_i$ be player i's information partition of Ω. For $\omega \in \Omega$, $Q_i(\omega)$ denotes the element of $\mathcal{Q}_i$ that contains ω. An event $E \subseteq \Omega$ is $\mathbf{p}$-believed at $\omega \in \Omega$ if each player i believes E at least with probability p_i conditional on $Q_i(\omega)$, that is, $P[E|Q_i(\omega)] \geq p_i$ for every i. Let $B_*^{\mathbf{p}}(E)$ be the set of states at which E is $\mathbf{p}$-believed. An event E is $\mathbf{p}$-evident if $E \subseteq B_*^{\mathbf{p}}(E)$, that is, if it is $\mathbf{p}$-believed whenever it is true. Kajii and Morris (1997) show the *critical path result*: if $\sum_i p_i < 1$, then for any event E with high ex ante probability, there exists a $\mathbf{p}$-evident subset F of E with high ex ante probability. Take E as the event that payoffs are given by those in the complete information game. Since the incomplete information game is close enough to the complete information game, $P[E]$ is close enough to one. By the critical path result, there exists a $\mathbf{p}$-evident subset F of E such that $P[F]$ is close enough to one. Then Kajii and Morris consider a restricted set of strategies in which each player i plays a_i^* at any state in F. By the construction of F and the definition of $\mathbf{p}$-dominance, taking a_i^* in F is a best response for player i if player j takes a_j^* in F for every $j \neq i$. Therefore the best response correspondence defined on this restricted strategy set is nonempty-valued. Thus there is a fixed point of this best response correspondence, which is an equilibrium in the incomplete information game. In this equilibrium, the ex ante probability that agents play a^* in the equilibrium is close enough to one.

The two proofs are similar to each other. The product of intervals, $\prod_{i=1}^{n}[T_i, \infty)$, in our proof has the property that after T_i, every agent in population i puts probability at least p_i that her opponents' most recent revisions occured within this set. This corresponds to the $\mathbf{p}$-evident set F in Kajii and Morris' proof. In both proofs, the best response correspondence defined on the restricted strategy set is nonempty-valued, and hence has a fixed point, which corresponds to an equilibrium of the incomplete information game and a perfect foresight path under the perfect foresight dynamics, respectively. Actually, the parallelism between the robustness approach and our result is more than a coincidence. As Takahashi (2008) points out, the perfect foresight dynamics can be regarded as a static incomplete information game where player i's type in the incomplete information game corresponds to time when an agent in population i receives a revision opportunity. A strategy profile is a Bayesian Nash equilibrium in the incomplete information game if and only if it induces a perfect foresight path in the perfect foresight dynamics.

Despite the aforementioned similarity, there is an important difference between our approach and Kajii and Morris (1997). In Kajii and Morris (1997), the critical path result implies that a $\mathbf{p}$-evident subset of a high ex ante probability event has a high ex ante probability, whereas our corresponding result shows the existence of finite $\mathbf{T}$ but is silent about the "ex ante probability" of the set $\prod_i[T_i, \infty)$. This is because, as Takahashi points out, in the perfect foresight dynamics interpreted as an incomplete information game, "types" are distributed according to an improper distribution. Therefore, it is impossible to obtain a result concerning the ex ante probability of $\prod_i[T_i, \infty)$, and we only have that this set is "large enough" in the sense that each T_i is finite. We introduce a technique of random ordering to cope with this difference. As explained in Section 3.1, we can associate random ordering with beliefs under the perfect foresight dynamics with zero subjective discount rates. This fact and the simple structure of random ordering enable us to find $\mathbf{T}$.

Our method of proof derives slightly stronger results than existing ones. We establish our results for not only positive but also zero and negative subjective discount rates. We also allow subjective discount rates and revision speeds to be heterogenous among populations.

We also investigate several issues concerning robustness and extensions of this result. First we consider rationalizable foresight dynamics defined by Matsui and Oyama (2006), which relaxes the perfect foresight assumption while retaining common knowledge of rationality. We show that a strict $\mathbf{p}$-dominant equilibrium is globally accessible and absorbing under rationalizable foresight dynamics. Then we consider the concept of (strict) $\mathbf{p}$-best response