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A comparison of integer and constraint programming models for the deficiency problem

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ABSTRACT

An edge-coloring of a graph $G=(V,E)$ is a function c that assigns an integer $c(e)$ (called color) in $\{0,1,2,\ldots\}$ to every edge $e\in E$ so that adjacent edges are assigned different colors. An edge-coloring is compact if the colors of the edges incident to every vertex form a set of consecutive integers. The deficiency problem is to determine the minimum number of pendant edges that must be added to a graph such that the resulting graph admits a compact edge-coloring. We propose and analyze three integer programming models and one constraint programming model for the deficiency problem.

1. Introduction

All graphs considered in this paper are connected, have no loops, but may contain parallel edges. An *edge-coloring* of a graph $G=(V,E)$ is a function $c:E\rightarrow\{0,1,2,\ldots\}$ that assigns a color $c(e)$ to every edge $e\in E$ such that $c(e)\neq c(e')$ whenever e and e' share a common endvertex. A *k-edge-coloring* is a similar function, but uses only colors in $\{0,1,\ldots,k-1\}$. Let E_v denote the set of edges incident to vertex $v\in V$. The *degree* $\deg(v)$ of a vertex v is the number of edges in E_v and the maximum degree in G is denoted $\Delta(G)$. Note that all k-edge-colorings of a graph G use at least $\Delta(G)$ different colors, which means that $\Delta(G)\leq k$.

An edge-coloring of a graph $G=(V,E)$ is *compact* if $\{c(e):e\in E_v\}$ is a set of consecutive nonnegative integers for all vertices $v\in V$. The terms *consecutive edge-colorings* [4,7] and *interval edge-colorings* [1,2,8,9,11,13] are also used by some authors. A graph is *compactly colorable* if it admits a compact edge-coloring. For an edge-coloring c of a graph $G=(V,E)$, let $\underline{c}(v)=\min_{e\in E_v}\{c(e)\}$ and $\overline{c}(v)=\max_{e\in E_v}\{c(e)\}$ denote, respectively, the smallest and the largest color assigned to an edge incident to v. It follows from the above definitions that if c is compact, then $\overline{c}(v)=\underline{c}(v)+\deg(v)-1$ for all vertices $v\in V$.

The problem of determining a compact k-edge-coloring (if any) of a graph was introduced by Asratian and Kamalian [2]. Determining whether or not a given graph is compactly colorable is known to be an $\mathcal{NP}-$complete problem [13], even for bipartite graphs. Given a k-edge-coloring c of a graph G and a vertex v, the *deficiency of c at v*, denoted $d_v(G,c)$, is the minimum number of integers that must be added to $\{c(e):e\in E_v\}$ to form a set of consecutive integers. The *deficiency* of c is then defined as the sum $d(G,c)=\sum_{v\in V}d_v(G,c)$. Hence, c is compact if and only if $d(G,c)=0$. The *deficiency of a graph G*, denoted $d(G)$, is the minimum deficiency $d(G,c)$ over all edge-colorings c of G. This concept, which was introduced by Giaro et al. [5], provides a measure of how close G is to be compactly colorable. Indeed, $d(G)$ is the minimum number of pendant edges that must be added to G such that the resulting graph is compactly colorable. The problem of determining the deficiency of a graph is $\mathcal{NP}-$hard [4]. This problem is also studied in [1,3,5,7–9,11,12].

Vizing's theorem [14] guarantees the existence of a k-edge-coloring for all $k\geq\Delta(G)+1$. But it may happen that $d(G,c)=d(G)$ only if c uses strictly more than $\Delta(G)+1$ colors. For example, it is not difficult to verify that the clique K_5 on five vertices has no edge-coloring with $\Delta(K_5)=4$ colors, and that all 5-edge-colorings c of K_5 have a deficiency $d(K_5,c)=3$. However, as illustrated in Fig. 1, it is not difficult to color the edges of K_5 with six colors and deficiency of 2.

The problem of determining a compact k-coloring of a graph often arises in scheduling problems with compactness constraints [6]. For example, the open shop problem considers m processors $P_1,\ldots,P_m$ and n jobs $J_1,\ldots,J_n$. Each job J_i is a set of s_i tasks. Suppose that each task has to be processed in one time unit on a specific processor. No two tasks of the same job can be processed simultaneously and no processor can work on two tasks at the same time. Moreover, compactness requirements state that waiting periods are forbidden for every job and no idles are allowed on any processor. In other words, the time periods assigned to the tasks of a job must be consecutive, and each processor must be active

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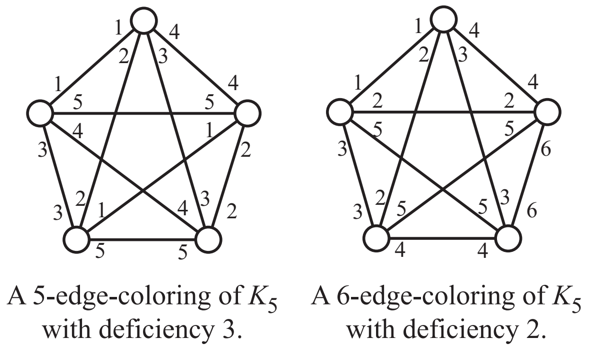

Fig. 1. The minimum deficiency of K_5 can only be achieved by using at least $\Delta(G)+2$ colors.

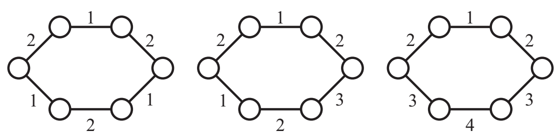

Fig. 2. k-edge-colorings of C_6 with $k=2,3,4$.

during a set of consecutive periods. The existence of a feasible compact schedule with k time periods is equivalent to the existence of a compact k-edge-coloring of the graph G that contains one vertex for each job and each processor, and one edge for each task (i.e., a task of job J_i to be processed on P_j is represented by an edge between the vertices representing J_i and P_j). Each color used in the k-edge-coloring corresponds to a time period. The compactness requirements for each job and each processor are equivalent to imposing that the colors appearing on the edges of E_v must be consecutive for every vertex v in G. If the waiting periods of the jobs and the idles on the processors are not forbidden but their number has to be minimized, the problem is then to find an edge-coloring of G with minimum deficiency.

In this paper, we compare different models for computing the deficiency $d(G)$ of a graph G. In Section 2, we give an upper bound on the number of colors used in an edge-coloring with minimum deficiency. This bound is used to reduce the number of variables in the various models presented in Section 3. The performances of the proposed models are compared in Section 4.

2. An upper bound on the number of colors

Let $s(G)$ be the smallest integer such that G admits an $s(G)$-edge-coloring c with deficiency $d(G,c)=d(G)$. Similarly, let $S(G)$ be the largest integer such that G admits an $S(G)$-edge-coloring c with deficiency $d(G,c)=d(G)$. Note that if G is compactly colorable, then these definitions correspond to those in [7]. We clearly have $\Delta(G)\le s(G)\le S(G)$. For example, it is not difficult to show that for a chordless cycle C_6 on six vertices, we have $s(G)=2$ and $S(G)=4$. k-edge-colorings of C_6 with deficiency $d(C_6)=0$ are shown in Fig. 2 for $k=2,3,4$. Note that a graph G does not necessarily admit a k-edge-coloring with deficiency $d(G)$ for all values of $k\in\{s(G),\dots,S(G)\}$. For example, Sevastianov [13] has given a graph G with $s(G)=100$, $S(G)=173$, and for which there is no k-edge-coloring with minimum deficiency when $k\in\{101,\dots,172\}$.

Giaro et al. [7] have proved that if a graph G is compactly colorable (i.e., $d(G)=0$), then $S(G)\le 2n-4$, where n is the number of vertices in G. We extend this result to all graphs G, showing that $S(G)\le 2n-4+d(G)$. In their proof, Giaro et al. use the notions of e-paths, i-hairs and i-nodes defined on a graph G' obtained from G by adding two vertices and two edges. We define these notions in a different but equivalent way, since we prefer to work directly with G instead of its augmented graph G'.

For an edge-coloring c, we denote $c_{\min}$ and $c_{\max}$ the minimum and maximum color used in c. Also, we denote $V_c^{\min}$ (respectively $V_c^{\max}$) the subset of vertices incident to an edge having color $c_{\min}$ (respectively $c_{\max}$). An e-path P for c is a simple path with vertex set $\{v_1,\dots,v_p\}$, $p\ge 1$ such that $v_1\in V_c^{\min}$, $v_p\in V_c^{\max}$, and every v_i is adjacent to v_{i+1} $(1\le i<p)$. We denote by e_i the edge with endvertices v_i and v_{i+1}. Moreover:

- an i-hair $(1\le i\le p)$ of P is an edge e incident to v_i and such that
  - $c(e_{i-1})<c(e)<c(e_i)$ if $1<i<p$,
  - $c(e)<c(e_1)$ if $i=1$,
  - $c(e)>c(e_{p-1})$ if $i=p$;
- an i-node $(1\le i\le p)$ is the endvertex of an i-hair other than v_i.

Let V_P and E_P be the vertex set and the edge set of an e-path P, and let W_P be its set of i-nodes and H_P its set of i-hairs. The skeleton of P is the subgraph G_P of G with vertex set $V_P\cup W_P$ and edge set $E_P\cup H_P$. Clearly, if $d(G,c)=0$, then the skeleton of every e-path P for c contains at least one edge of color k for each $k\in\{c_{\min},\dots,c_{\max}\}$. Giaro et al. [7] have proved that if c is an edge-coloring of G with $d(G,c)=d(G)=0$, then G contains an e-path P for c such that the degree of every i-node in its skeleton G_P is 1 or 2. We are now ready to prove the upper bound on $S(G)$, using a similar proof as in [7].

Theorem 1. *If G is a graph with $n\ge 3$ vertices, then*

$S(G)\le 2n-4+d(G)$.

Proof. Let c be an edge-coloring of G that uses $S(G)$ colors and such that $d(G,c)=d(G)$. We augment G to $\tilde{G}$ by introducing $d_v(G,c)$ pendant edges to each deficient vertex v. Hence, a total of $d(G)$ new vertices and $d(G)$ new edges are added to G, and we can now extend c to a compact $S(G)$-edge-coloring $\tilde{c}$ of $\tilde{G}$ by assigning the missing colors to the new edges around each vertex v.

Consider an e-path P for $\tilde{c}$ such that the degree of every i-node in its skeleton $\tilde{G}_P$ is 1 or 2. Let R be the set of vertices in $\tilde{G}_P$ that do not belong to G, and let I denote the set of i-nodes of $\tilde{G}_P$ that do not belong to R. Let p denote the number of vertices in P and let m be the number of edges in $\tilde{G}_P$. The number of vertices in $\tilde{G}_P$ is then equal to $p+|I|+|R|\le n+|R|$.

Note that the i-nodes of I can be of degree 1 or 2 in $\tilde{G}_P$, but those in R are of degree 1. Since $\tilde{c}$ uses $S(G)$ colors which all appear on the edges of $\tilde{G}_P$, we have $S(G)\le m$, where m is the number of edges in $\tilde{G}_P$. Hence, it is sufficient to prove that m is not larger than $2n-4+d(G)$.

If $p=1$, then $\tilde{G}_P$ is a star and $m=|I|+|R|\le(n-1)+d(G)\le 2n-4+d(G)$. So assume $p\ge 2$. We then have

$$m\le 2|I|+|R|+p-1\le 2n-(p+1)+|R|\le 2n-(p+1)+d(G).$$

If $p\ge 3$ then $m\le 2n-4+d(G)$. So assume $p=2$ while $m\ge 2n-3+d(G)$. Then the above inequalities become equalities. Consequently, each i-node in I has degree 2 in $\tilde{G}_P$ and $n=|I|+2$. Hence, $\tilde{G}$ and $\tilde{G}_P$ have the same vertex set, and $n\ge 3$ implies $|I|>0$. It follows that the $|I|$ largest possible colors for the 1-hairs that link v_1 to the vertices in I are $\tilde{c}(e_1)-|I|,\dots,\tilde{c}(e_1)-1$, while the $|I|$ smallest possible colors for the 2-hairs that link v_2 to the vertices in I are $\tilde{c}(e_1)+1,\dots,\tilde{c}(e_1)+|I|$. Since $\tilde{c}$ is compact while every vertex in R has degree 1, $\tilde{G}$ necessarily contains edges that link several pairs of vertices in I. The number of such edges is at least equal to

$$\frac{1}{2}\left(\sum_{j=1}^{|I|}(c(e_1)+j)-\sum_{j=1}^{|I|}(c(e_1)-j)-\sum_{j=1}^{|I|}1\right)=\frac{|I|^2}{2}$$

But the number of such edges cannot be larger than $|I|(|I|-1)/2$, a contradiction.□