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Discrete Optimization

# Robust mixed-integer linear programming models for the irregular strip packing problem

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ABSTRACT

Two-dimensional irregular strip packing problems are cutting and packing problems where small pieces have to be cut from a larger object, involving a non-trivial handling of geometry. Increasingly sophisticated and complex heuristic approaches have been developed to address these problems but, despite the apparently good quality of the solutions, there is no guarantee of optimality. Therefore, mixed-integer linear programming (MIP) models started to be developed. However, these models are heavily limited by the complexity of the geometry handling algorithms needed for the piece non-overlapping constraints. This led to pieces simplifications to specialize the developed mathematical models. In this paper, to overcome these limitations, two robust MIP models are proposed. In the first model (*DTM*) the non-overlapping constraints are stated based on direct trigonometry, while in the second model (*NFP – CM*) pieces are first decomposed into convex parts and then the non-overlapping constraints are written based on nofit polygons of the convex parts. Both approaches are robust in terms of the type of geometries they can address, considering any kind of non-convex polygon with or without holes. They are also simpler to implement than previous models. This simplicity allowed to consider, for the first time, a variant of the models that deals with piece rotations. Computational experiments with benchmark instances show that *NFP – CM* outperforms both *DTM* and the best exact model published in the literature. New real-world based instances with more complex geometries are proposed and used to verify the robustness of the new models.



## 1. Introduction

Two-dimensional (2D) irregular packing problems, also known as nesting problems, are cutting and packing problems in which, as in all cutting and packing problems, small items or pieces have to be cut from a larger object (or packed inside a larger object) so that the unused regions of the large object, usually designated as waste, are minimized. According to Bennell and Oliveira (2008), what differentiates irregular packing problems from the other cutting and packing problems is that the pieces are not regular, i.e. a non-trivial handling of geometry is involved. In the 2D irregular strip packing problem, only two dimensions of the material to cut are relevant for planning purposes. The goal is to cut all pieces while minimizing the used length of the large object. According to Wäscher, Haußner, and Schumann (2007) this is an input minimization problem and, within this category of problems, it can be classified as 2D Open Dimension Problem. These problems derive from rather important applications in, among others, the textile, garment, furniture, footwear and metalwork industries, being economically very relevant.

The scientific community has paid a lot of attention to irregular packing problems and rather complex and sophisticated heuristic methods have been developed and recently published. Good examples of fairly recent and high performance approaches are: 2DNEST (Egeblad, Nielsen, & Odgaard, 2007), ILS(QN) (Imamichi, Yagiura, & Nagamochi, 2009), FITS (Umetani et al., 2009), ELS (Leung, Lin, & Zhang, 2012) and SA-SMT (Sato, Martins, & Tsuzuki, 2012). They are all rather complex algorithms, based on metaheuristics and with more than one level of search, incorporating nonlinear programming models, constructive heuristics and local search methods.

However, the best heuristic method known up to now was proposed by Elkeran (2013). It starts with a pairwise clustering algorithm, to group congruent polygons together in pairs, followed by a guided cuckoo search with two phases: the first one is responsible for minimizing the board length whilst the second one handles the overlap minimization problem (as in several previously presented

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methods). The cuckoo search itself is a populational metaheuristic incorporating a local random walk and a global explorative random walk using Lévy flights. With the cuckoo search in charge of moving a piece inside the board to a less overlapping position, a guided local search algorithm is used to escape local minima.

The heuristic methods present an increasingly higher implementation complexity and, despite the increasingly better results obtained for commonly used benchmark instances, by nature they are not able to prove solution optimality or even to provide any measure of distance to the optimal solution. This theoretical drawback led to an increasing interest of researchers on exact methods, namely based on mathematical programming models, for irregular packing problems. To write mathematical programming models it is needed to write constraints ensuring that pieces do not overlap and are completely contained inside the board. Baldacci, Boschetti, Ganovelli, and Maniezzo (2014) use a raster approach to deal with the geometric constraints and discretizes both the large object and the pieces into a bit-matrix and imposes a cover constraint to each position of the matrix. The accuracy of this model depends on the resolution of the bit-matrix. Actually, Baldacci et al. do not deal with the strip packing problem but with an irregular multiple stock-size cutting stock problem, reason why the published results are not comparable with the other cited methods.

In what concerns the strip packing problems, Toledo, Carravilla, Ribeiro, Oliveira, and Gomes (2013) proposed the Dotted-Board Model where the placement points on the board are restricted to a set of integer positions. Binary variables are assigned to the placement of piece types on a given point of the board and this model proved to be rather efficient being able to solve to optimality instances till 21 pieces of 7 different types and 56 pieces of just 2 different types. However, it should be noticed that these are optimal solutions in what concerns the grid used to discretize the board. Continuous placement models do not have this drawback but are able to optimally solve significantly smaller instances. The three published models that use continuous decision variables for the placement of the pieces resort to the concept of nofit polygon to write the non-overlapping constraints. The nofit polygon is a polygon derived from aggregating two component polygons that represents the pieces and that captures the geometric characteristics of both pieces (Bennell & Oliveira, 2008), transforming the analysis of the relative position of two pieces in the much simpler analysis of the relative position of one point and one polygon. From a mathematical programming point of view, the feasible placement region of a given piece is the set of points that are in the exterior or on the frontier of the nofit polygons of that piece and all other remaining pieces. Focusing the attention in just a pair of pieces, the difference of the three models is on how the exterior of the nofit polygon is described. Therefore Carravilla and Ribeiro (2005) and Gomes and Oliveira (2006) use a set covering constraint to describe the exterior of the nofit polygon and Fischetti and Luzzi (2009) and Alvarez-Valdes, Martinez, and Tamarit (2013) use a set partitioning constraint. They differ on how the partitions are created, being the horizontal slices approach of Alvarez-Valdes et al. (2013) the best continuous exact approach known up to date. With this model it was possible to solve to optimality instances to up to 16 pieces in 5 hours of computation time.

The complexity of the mathematical programming models depends heavily on how the geometry layer of the problem is dealt with. In practice, all the models impose constraints to the pieces or board geometry, either by discretizing the pieces and/or the board, or by considering simplified piece geometries, as dealing only with convex pieces or not allowing holes in the pieces, pushing the models away from the real-world needs. Part of the models' limitations come from the limitations of the algorithms available to build the nofit polygons (Bennell & Oliveira, 2008). If, for pieces represented by convex polygons, building the nofit polygon is just a matter of ordering the edge angles, when non-convex pieces are present both the sliding algorithms and Minkowski sums algorithms face hard, sometimes insurmountable obstacles like narrow entries, similar edge slopes and piece holes. This leads to time consuming, complex and numerically unstable algorithms.

In this paper, to overcome these limitations and produce robust mathematical programming models, two directions are proposed. The first one does not use the nofit polygon but instead derives non-overlapping constraints based on direct trigonometry (Bennell & Oliveira, 2008). The second one decomposes the pieces into convex parts and then computes nofit polygons for the convex parts. The two approaches are robust both in terms of numeric stability and on the type of geometries they can address, considering any kind of non-convex piece with any type of holes inside them.

The remainder of this paper is organized as follows. After this first introductory section, Section 2 presents the basic geometric definitions used along the paper. In Section 3 the two new mathematical programming models are presented, including some variants based on valid inequalities and variable reduction strategies, and in Section 4 computational experiments on benchmark instances commonly used in the literature are presented. The first computational experiments aim at comparing the new models and their variants and the second experiments aim at comparing the best new model against the best model published in the literature. In a second phase the computational experiments focus on new geometrically more complex instances, aiming to prove the flexibility and robustness of the proposed models.

## 2. Problem description and geometric definitions

This section presents a formal definition of the irregular strip packing problem and the geometric definitions that will be used along this paper.

### 2.1. Problem definition

The two-dimensional irregular strip packing problem can be formally defined by a set of pieces of $m$ distinct types, each one described by a polygon $P_i$, $i = 1, \ldots, m$, that have to be placed, in an amount of $d_i$ units, on a large rectangle board characterized by a height $H$ and a length $L$. For the sake of legibility and comparability the models will be presented and tested considering that piece rotations are not allowed, as in all previous exact approaches to this problem. However, an extension of the models allowing different orientations for each piece will be proposed and discussed in Section 3.3. As the length $L$ is not fixed, the board can be considered as having "infinite" length and the problem's goal is to minimize $L$, i.e. the length of the board that is necessary to cut all demanded pieces. In practice, this corresponds to minimizing the amount of raw-material used to satisfy a given order of irregularly shaped pieces.

The problem constraints are of three types:

1. each piece type $i$ has to be cut in the demanded quantities $d_i$, in a total of $N = \sum_{i=1}^{m} d_i$ pieces;
2. pieces must not overlap, i.e. $\mathrm{int}(P_j) \bigcap \mathrm{int}(P_l) = \emptyset, \ \forall j, l = 1, \ldots, N;\ j \neq l$;
3. pieces must be completely contained inside the board, i.e.: $\mathrm{int}(P_j) \bigcap \mathrm{int}(\mathrm{rect}(L, H)) = \mathrm{int}(P_j), \ \forall j = 1, \ldots, N$;

where int () stands for the topological interior and rect($L$,$H$) for the rectangle of length $L$ and height $H$.

### 2.2. Basic concepts

In the new models proposed in this paper each piece $i$ is not described by a single polygon but it is decomposed and