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A representation of L-domains by information systems

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ABSTRACT

In this paper, we propose the notion of L-information systems which provides a concrete representation of L-domains. In particular, we prove that the category of L-information systems with approximable mappings as morphisms is equivalent to that of L-domains with Scott continuous functions as morphisms.

1. Introduction

Information systems are applied in computer science as an alternative for the usual domain-theoretical approach to the semantics of programming languages. In 1982, Scott [7] proposed the notion of information systems and provided a logic-oriented approach to denotational semantics of programming languages. Later on, Larsen and Winskel [17] proved that the category of such type of information systems is equivalent to that of algebraic Scott domains with continuous functions as morphisms. Much effort has been made on the representation of other domains. For instance, Winskel [15] developed the notion of stable event structures as a model for processes of concurrent computation, which provides a concrete representation of DI-domains. Zhang [11,12] proposed a special types of information systems which realize the presentation of DI-domain and SFP-domain respectively. Hoofman [18] generalized the (Scott) information systems to the continuous case and obtained the representation of bounded complete domains. The continuous information systems which exactly capture continuous domains were introduced by Spreen et al. [8]. More articles about information systems can be referred to [4,5,16,20].

A well-known fact in Domain theory is that continuous domains and Scott continuous functions do not form a Cartesian closed category, but the category of L-domains is a maximal Cartesian closed full subcategory of continuous domains [1,2]. In 1992, Zhang [13] developed a logic-oriented approach to representing algebraic L-domains by Gentzen-style systems which is not Scott-style. In 2012, Spreen [9] proposed a notion of L-information systems which exactly captured L-domains (actually pointed L-domains). However, the axioms used in the definition of L-information systems therein are much complex, which makes it not easy for its further applications. Our interest is to provide a more simplified form of information systems which can serve as a representation of L-domains. The main result of this paper is: by adding new axioms to the

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information systems in the sense of Spreen et al. [8], we propose the notion of L-information systems, which provides a concrete representation of L-domains. In particular, we show that the category of L-domains with Scott continuous functions is equivalent to that of L-information systems with approximable mappings.

The paper is organized as follows: In Section 2, we recall some basic notions of domain theory and information systems. In Section 3, we present the notion of L-information systems and investigate the relationship between L-information systems and L-domains. In Section 4, the equivalence between the category of L-information systems and that of L-domains is obtained.

2. Domains and continuous information systems

In this paper, for any set A, we write $F \sqsubseteq A$ to mean that F is a finite subset of A. Let $(P, \leq)$ be a poset. For any $x \in P$, we use $\downarrow x$ to denote the principal ideal determined by x, i.e., $\downarrow x = \{p \in P \mid p \leq x\}$. A nonempty subset D of P is said to be *directed* if for arbitrary elements $a, b \in D$, there exists $c \in D$ such that $a \leq c$ and $b \leq c$. We use $\bigsqcup D$ to denote the least upper bound of a directed subset D. A poset is called a *dcpo* if every directed subset has a least upper bound. Given $x,\ y \in P$, we say x is *way below* y (in symbol $x \ll y$) if for any directed subset $D \subseteq P$ with $\bigsqcup D$ exists, $y \leq \bigsqcup D$ always implies the existence of $d \in D$ with $x \leq d$. For any $x \in P$, we use $\Downarrow x$ to denote the set $\{a \in P \mid a \ll x\}$. A subset $B \subseteq P$ is called a *basis* of P if for every $x \in P$, $\Downarrow x \cap B$ is a directed subset and $x = \bigsqcup(\Downarrow x \cap B)$. A dcpo is called a *domain* if it has a basis.

Lemma 2.1. *(See [10], Proposition O-2.2.) Let P be a poset. For P to be a compete lattice, it is sufficient to assume the existence of sups of any finite set and any directed set.*

Lemma 2.2. *Let P be a dcpo. For any element $a \in P$, there exists a maximal element b of P such that $a \leq b$. Moreover, if P is a domain, then $\Downarrow M(P)$ is a basis of P, where $M(P)$ is the set of all maximal elements of P.*

Proof. Denote $\mathscr{C}$ as the set of all chains of P containing a. By the Hausdorff maximal principle, there exists a maximal element C in $\mathscr{C}$. Since P is a dcpo, then the least upper bound of C exists which is denoted as b. Obviously, b is a maximal element of P and $a \leq b$.

Suppose P is a domain. Then for any $a \in P$, $\Downarrow a$ is directed and $a = \bigsqcup \Downarrow a$. By the above proof, there exists $b \in M(P)$ such that $a \leq b$. Then we have $\Downarrow a \subseteq \Downarrow b \subseteq \Downarrow M(P)$ and $\Downarrow a = \Downarrow a \cap \Downarrow M(P)$. Therefore, $\Downarrow M(P)$ is a basis of P. □

Lemma 2.3. *Let P be a domain. For any element $x \in P$ and directed set $D \subseteq P$, if $x \ll \bigsqcup D$, then there exists $d \in D$ such that $x \ll d$.*

Proof. Since P is a domain, we have $d = \bigsqcup \Downarrow d$ for any $d \in D$. Set $D' = \{a \mid a \in \Downarrow d$ for some $d \in D\}$, then D' is directed and $\bigsqcup D' = \bigsqcup D$. Because $x \ll \bigsqcup D$, there exists $a \in D'$ such that $x \leq a$. This implies that there exists $d \in D$ such that $x \ll d$. □

Definition 2.1. A function $\varphi : P \to P'$ between dcpos is said to be *Scott continuous* if it is monotone and preserves the least upper bounds of directed sets, i.e., for any directed subset D of P,

$$\varphi(\bigsqcup D) = \bigsqcup \varphi(D).$$

Definition 2.2. A domain in which every principal ideal is a complete lattice (in its induced order) is called an *L-domain*. An L-domain with a least element is called a *pointed L-domain.*

In the following, we use **DOM** and **LDOM** to denote the categories of domains and L-domains respectively (both with Scott continuous functions as morphisms). Standard references for Domain theory are [3,6,10,14].

Next, we recall the notion of *equivalence of categories* which is used to demonstrate strong similarities between mathematical structures.

Given a category **C**, the sets of its objects and morphisms (also called arrows) are denoted as $\mathbf{C}_o$ and $\mathbf{C}_a$, respectively. A morphism $f : A \to B$ is called an *isomorphism* if it has an inverse $f^{-1} : B \to A$. Let **C** and **D** be categories. A *functor* $\mathfrak{F} : \mathbf{C} \to \mathbf{D}$ is a pair of maps $\mathfrak{F}_o : \mathbf{C}_o \to \mathbf{D}_o$ and $\mathfrak{F}_a : \mathbf{C}_a \to \mathbf{D}_a$ which satisfy: (1) if $f : A \to B$ is a morphism in **C**, then $\mathfrak{F}_a(f) : \mathfrak{F}_o(A) \to \mathfrak{F}_o(B)$ is a morphism in **D**; (2) for any object A of **C**, $\mathfrak{F}_a(\mathrm{id}_A) = \mathrm{id}_{\mathfrak{F}_o(A)}$; (3) if $g \circ f$ is defined in **C**, then $\mathfrak{F}_a(g) \circ \mathfrak{F}_a(f)$ is defined in **D** and $\mathfrak{F}_a(g \circ f) = \mathfrak{F}_a(g) \circ \mathfrak{F}_a(f)$.

Definition 2.3. The categories **C** and **D** are *equivalent* if there are

(1) a functor $\mathfrak{F} : \mathbf{C} \to \mathbf{D}$ and a functor $\mathfrak{G} : \mathbf{D} \to \mathbf{C}$,
(2) a family of isomorphisms $\{\mu_C : C \to \mathfrak{G}_o(\mathfrak{F}_o(C)) \mid C \in \mathbf{C}_o\}$ with the property that for every morphism $f : C \to C'$ of **C**, $\mathfrak{G}_a(\mathfrak{F}_a(f)) \circ \mu_C = \mu_{C'} \circ f$,