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A fast algorithm for the linear canonical transform

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ABSTRACT

In recent years there has been a renewed interest in finding fast algorithms to compute accurately the linear canonical transform (LCT) of a given function. This is driven by the large number of applications of the LCT in optics and signal processing. The well-known integral transforms: Fourier, fractional Fourier, bilateral Laplace and Fresnel transforms are special cases of the LCT. In this paper we obtain an $\mathcal{O}(N\log N)$ algorithm to compute the LCT by using a chirp-FFT-chirp transformation yielded by a convergent quadrature formula for the fractional Fourier transform. This formula gives a unitary discrete LCT in closed form. In the case of the fractional Fourier transform the algorithm computes this transform for arbitrary complex values inside the unitary circle and not only at the boundary. This chirp-FFT-chirp transform approximates the ordinary Fourier transform more precisely than just the FFT, since it comes from a convergent procedure for non-periodic functions.

1. Introduction

The linear canonical transform (LCT) of a given function $f(x)$ is a three-parameter integral transform that was obtained in connection with canonical transformations in Quantum Mechanics [1,2]. It is defined by

$$\mathcal{L}^{\{a,b,c,d\}}[f(x),y] = \frac{1}{\sqrt{2\pi i b}}\int_{-\infty}^{\infty} e^{i/2b(ax^2-2xy+dy^2)} f(x)\, dx,$$

for $b \neq 0$, and by $\sqrt{d}e^{(i/2)cdy^2} f(dy)$, if $b=0$. The four parameters a, b, c and d appearing in the above expression, are the elements of a 2×2 matrix with unit determinant, i.e., $ad-bc=1$. Therefore, only three parameters are free. Since this transform is a useful tool for signal processing and optical analysis, its study and direct computation in digital computers have become an important issue [3–10], particularly, fast algorithms to compute the linear canonical transform have been devised [4,7]. These algorithms use the following related ideas: (a) use of the periodicity and shifting properties of the discrete LCT to break down the original matrix into smaller matrices as the FFT does with the DFT, (b) decomposition of the LCT into a chirp-FFT-scaling transformation and (c) decomposition of the LCT into a fractional Fourier transform followed by a scaling-chirp multiplication. All of these are algorithms of $\mathcal{O}(N\log N)$ complexity.

In this paper we present an algorithm that takes $\mathcal{O}(N\log N)$ time based in the decomposition of the LCT into a scaling-chirp-DFT-chirp-scaling transformation, obtained by using a quadrature formula of the continuous Fourier transform [11,12]. Here, DFT stands for the standard discrete Fourier transform. To distinguish this discretization from other implementations, we call it the extended Fourier Transform (XFT). Thus, the quadrature from which the XFT is obtained, uses some asymptotic properties of the Hermite polynomials and yields a fast algorithm to compute the Fourier transform, the fractional Fourier transform and therefore, the LCT. The quadrature formula is $\mathcal{O}(1/N)$–convergent to the continuous Fourier transform for certain class of functions [13].

2. A discrete fractional Fourier transform

In previous work [12–14], we derived a quadrature formula for the continuous Fourier transform which yields

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an accurate discrete Fourier transform. For the sake of completeness we give in this section a brief review of the main steps to obtain this formula.

Let us consider the family of Hermite polynomials $H_n(x)$, n=0,1,…, which satisfies the recurrence equation:

$$H_{n+1}(x)+2nH_{n-1}(x)=2xH_n(x), \tag{1}$$

with $H_{-1}(x)\equiv 0$. Note that the recurrence equation (1) can be written as the eigenvalue problem

$$\begin{pmatrix} 0 & 1/2 & 0 & \cdots \\ 1 & 0 & 1/2 & \cdots \\ 0 & 2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} H_0(x) \\ H_1(x) \\ H_2(x) \\ \vdots \end{pmatrix}=x\begin{pmatrix} H_0(x) \\ H_1(x) \\ H_2(x) \\ \vdots \end{pmatrix}. \tag{2}$$

Let us now consider the eigenproblem associated to the principal submatrix of dimension N of (2)

$$\mathcal{H}=\begin{pmatrix} 0 & 1/2 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1/2 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1/2 \\ 0 & 0 & 0 & \cdots & N-1 & 0 \end{pmatrix}.$$

It is convenient to symmetrize $\mathcal{H}$ by using the similarity transformation $S\mathcal{H}S^{-1}$ where S is the diagonal matrix

$$S=\operatorname{diag}\left\{1,\frac{1}{\sqrt{2}},\ldots,\frac{1}{\sqrt{(N-1)!2^{N-1}}}\right\}.$$

Thus, the symmetric matrix $H=S\mathcal{H}S^{-1}$ takes the form

$$\begin{pmatrix} 0 & \sqrt{\frac{1}{2}} & 0 & \cdots & 0 & 0 \\ \sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{2}{2}} & \cdots & 0 & 0 \\ 0 & \sqrt{\frac{2}{2}} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \sqrt{\frac{N-1}{2}} \\ 0 & 0 & 0 & \cdots & \sqrt{\frac{N-1}{2}} & 0 \end{pmatrix}.$$

The recurrence equation (1) and the Christoffel–Darboux formula [15] can be used to solve the eigenproblem

$$Hu_k=x_ku_k,\quad k=1,2,\ldots,N,$$

which is a finite-dimensional version of (2). The eigenvalues x_k are the zeros of $H_N(x)$ and the kth eigenvector u_k is given by

$$c_k(s_1H_0(x_k),s_2H_1(x_k),\ldots,s_NH_{N-1}(x_k))^T,$$

where $s_1,\ldots,s_N$ are the diagonal elements of S and c_k is a normalization constant that can be determined from the condition $u_k^T\,u_k$=1, i.e., from

$$c_k^2\sum_{n=0}^{N-1}\frac{H_n(x_k)H_n(x_k)}{2^nn!}=1.$$

Therefore,

$$c_k=\sqrt{\frac{2^{N-1}(N-1)!}{N}\frac{(-1)^{N+k}}{H_{N-1}(x_k)}}.$$

Thus, the components of the orthonormal vectors u_k, k=1,2,…, N, are

$$(u_k)_n=(-1)^{N+k}\sqrt{\frac{2^{N-n}(N-1)!}{N(n-1)!}}\frac{H_{n-1}(x_k)}{H_{N-1}(x_k)}, \tag{3}$$

n=1,…,N. Let U be the orthogonal matrix whose kth column is u_k and let us define the matrix

$$\mathcal{F}_z=\sqrt{2\pi}U^{-1}D(z)U,$$

where $D(z)$ is the diagonal matrix $D(z)$=diag$\{1,z,z^2,\ldots,z^{N-1}\}$ and z is a complex number. Therefore, the components of $\mathcal{F}_z$ are given by

$$(\mathcal{F}_z)_{jk}=\sqrt{2\pi}\frac{(-1)^{j+k}2^{N-1}(N-1)!}{NH_{N-1}(x_j)H_{N-1}(x_k)}\sum_{n=0}^{N-1}\frac{z^n}{2^nn!}H_n(x_j)H_n(x_k). \tag{4}$$

Next, we want to prove that if N is large enough, (4) approaches the kernel of the fractional Fourier transform evaluated at x=x_j, y=x_k. To this, we use the asymptotic expression for $H_N(x)$ [15]

$$H_N(x)\simeq\frac{\Gamma(N+1)e^{x^2/2}}{\Gamma(N/2+1)}\cos\left(\sqrt{2N+1}\;x-\frac{N\pi}{2}\right). \tag{5}$$

Thus, the asymptotic form of the zeros of $H_N(x)$ are

$$x_k=\left(\frac{2k-N-1}{\sqrt{2N}}\right)\frac{\pi}{2}, \tag{6}$$

k=1,2,…,N. The use of (5) and (6) yields

$$H_{N-1}(x_k)\simeq(-1)^{N+k}\frac{\Gamma(N)}{\Gamma\left(\frac{N+1}{2}\right)}e^{x_k^2/2},\quad N\to\infty,$$

and the substitution of this asymptotic expression in (4) yields

$$(\mathcal{F}_z)_{jk}\simeq\sqrt{2\pi}\frac{2^{N-1}\left[\Gamma\left(\frac{N+1}{2}\right)\right]^2}{\Gamma(N+1)}e^{-(x_j^2+x_k^2)/2}\times\sum_{n=0}^{\infty}\frac{z^n}{2^nn!}H_n(x_j)H_n(x_k).$$

Finally, Stirling's formula and Mehler's formula [16] produce

$$(\mathcal{F}_z)_{jk}\simeq\sqrt{\frac{2}{1-z^2}}\exp\left[-\frac{(1+z^2)(x_j^2+x_k^2)-4x_jx_kz}{2(1-z^2)}\right]\Delta x_k, \tag{7}$$

where Δx_k is the difference between two consecutive asymptotic Hermite zeros, i.e.,

$$\Delta x_k=x_{k+1}-x_k=\frac{\pi}{\sqrt{2N}}. \tag{8}$$

Let us consider now the vector of samples of a given function $f(x)$

$$f=(f(x_1),f(x_2),\ldots,f(x_N))^T.$$