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A fast algorithm for finding the first intersection with a non-convex yield surface

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ABSTRACT

A major task in the numerical modelling of soils using complex elastoplastic material models is stress updating. This paper proposes a fast and robust numerical algorithm for locating the first intersection between a non-convex yield surface and an elastic trial stress path. The intersection problem is cast into a problem of finding the smallest positive root of a nonlinear function. Such a function may have multiple roots within the interval of interest. The method is based on the modified Steffensen method, with important modifications to address the issues arising from the non-convexity. Numerical examples demonstrate that the proposed M^2 Steffensen method is indeed computationally efficient and robust.

1. Introduction

Implementation of complex elastoplastic constitutive models for soils into finite element codes requires development of robust procedures for stress updating, the integration of the constitutive model. While many constitutive models lead to convex yield surfaces there are certain cases where the yield surface of a soil model is non-convex. For example, the yield surface for an unsaturated soil model is non-convex, if both saturated and unsaturated states of the soil are considered. Because both partial and full saturation are only different states of a soil, a single constitutive model should be expected to work for both states. As such, the non-convexity becomes inevitable. Wheeler et al. [13] pointed out the possible non-convexity of the most widely used model for unsaturated soils, the Barcelona Basic Model (BBM [1]) in the unsaturated zone. Sheng [7] argued that this non-convexity in BBM is inevitable if the pore water pressure is allowed to vary between positive and negative values. The non-convexity exists irrespective of the stress state variable used to formulate the constitutive model, as illustrated in Fig. 1. Some researchers have also argued that suction should be treated as a hardening (internal) variable instead of an additional variable in the stress space. However, this argument does not alter the fact that the size of the yield surface in stress space varies with suction and this variation does not necessarily result in plastic deformation. Another example of non-convexity appears when the Hvorslev envelope is added to the Mohr–Coulomb envelope in a Mohr–Coulomb type model. The Hvorslev envelope is usually used for low stress levels and is flatter than the Mohr–Coulomb envelope in the space of normal (or mean) stress versus shear (or deviator) stress, leading to a non-convex elastic zone around the transition.

A closely related problem arises when the elastic behaviour inside the yield surface is nonlinear. In this case, even if the yield surface is convex, the elastic trial stress path may cross the yield surface more than once. In other words, the elastic trial stress path is a curve instead of a straight line in the stress space. As such, a stress path that starts and ends inside the yield surface can still intersect with the yield surface (Fig. 2). In general, a convex elastic zone enclosing nonlinear behaviour is mathematically equivalent to a non-convex zone enclosing linear behaviour.

One of the main challenges in integrating constitutive models with non-convex yield surfaces or nonlinear elasticity is that the elastic trial stress path may cross the initial yield surface more than once, as illustrated in Fig. 1 for unsaturated soil models. Furthermore, the number of times the yield surface is crossed remains unknown, for a given strain increment and an initial stress state. In such circumstances, only the first intersection is of interest, as the stress path is likely to cause an initial elastoplastic loading followed by an elastic unloading. This leads to the key issue in solving non-convex models: to find the first intersection between the elastic trial stress path and the initial yield surface.

Integration of rate-type constitutive equations can be carried out in an implicit or explicit manner. In implicit schemes, all gradients and functions are evaluated at advanced unknown stress states and the solution is achieved by iteration. These methods do not usually involve a procedure to find the intersection between the elastic trial stress path and the yield surface. However, when the yield surface is non-convex, the knowledge of the first

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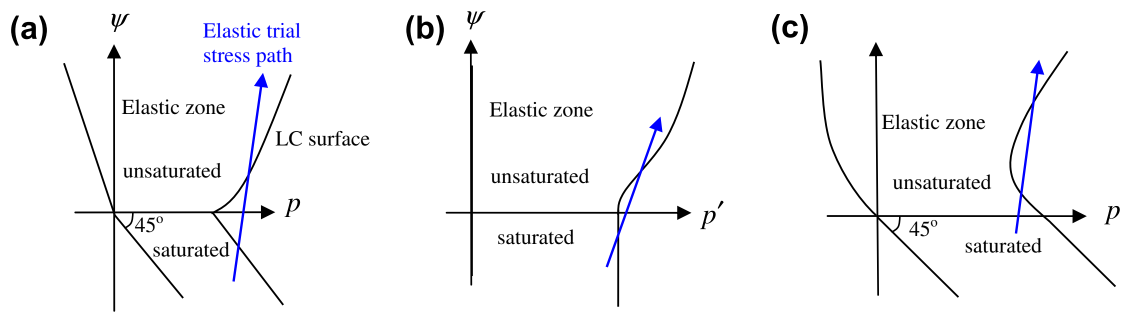

Fig. 1. Non-convexity of yield surfaces in unsaturated soil models in the suction–stress space (ψ: suction – pore air pressure in excess of pore water pressure, p and p': net and effective mean stress respectively; (a) net stress [1]; (b) effective stress [8]; and (c) net stress [9].

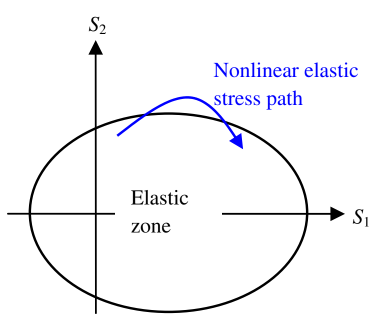

Fig. 2. Multiple intersection problem due to nonlinear elasticity (S_1 and S_2 are two stress variables).

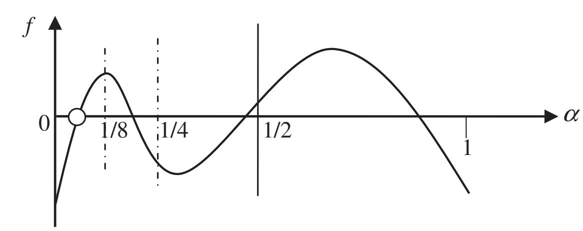

Fig. 3. Bracketing the roots for nonlinear function according to Pedroso et al. [6].

intersection seems to be a prerequisite for making these schemes work. Otherwise, a trial stress path that starts and ends in the initial elastic zone would be assumed to cause only elastic deformation. For unsaturated soil models, where the location of non-convexity is known, implicit schemes may work if the strain increments are kept sufficiently small [2]. The difficult question is: what is sufficiently small? On the other hand, explicit schemes estimate the gradients and functions at the current known stress states and proceed in an incremental fashion. These schemes invariably need to determine the intersection and substepping methods have been developed to control the integration error [11,12]. Therefore, it seems necessary for both implicit and explicit methods to find the first intersection between the current yield surface and the elastic trial stress path.

There is very little discussion in the literature about integrating non-convex soil models. Pedroso et al. [6] proposed a novel method for bracketing the intersection between the elastic trial stress path and a non-convex yield surface. Finding this intersection can be cast into a problem of finding the multiple roots of a nonlinear equation:

$$f(\alpha) = f(\mathbf{s}_\alpha, \mathbf{h}) = 0 \tag{1}$$

where $0 \leqslant \alpha \leqslant 1$, $f(\mathbf{s}, \mathbf{h})$ is the yield function, s is a set of external variables such as stress and suction, $\mathbf{h}$ is a set of internal variables, typically plastic strain or hardening parameters, subscript α indicates that the quantity is evaluated at strain increment $\alpha\Delta\boldsymbol{\varepsilon}$, the strain increment $\Delta\boldsymbol{\varepsilon}$ is assumed to be known, and the variable $\mathbf{s}_\alpha$ is assumed to be fully determined for given strain increment $\alpha\Delta\boldsymbol{\varepsilon}$, for example using elasticity theory.

The method proposed by Pedroso et al. [6] for bracketing the roots (α) is illustrated in Fig. 3. For a given increment ($\alpha = 1$), the number of roots of $f(\alpha)$ is first computed. If there is more than one root, the increment is divided into two equal sub-increments. The number of roots of each sub-increment is then computed. If the first sub-increment contains more than one root, it is further divided into two sub-increments. This process is repeated until the first sub-increment contains at most one root (Fig. 3). Once the roots are bracketed, the solution of the first root can be found by using numerical methods such as the Pegasus method [12].

Sheng et al. [10] applied the method by Pedroso et al. [6] to integrate an unsaturated soil model described by Sheng et al. [9] and found that the method can provide an accurate solution of the intersection problem. However, this approach was found to be computationally extremely expensive. It should be realised that the root-finding procedure must be applied for all strain increments at all Gauss points, irrespective of the starting and ending stress states. Pedroso's formula requires both the first and second orders of gradients of the yield function and also complex numerical integration to compute the number of roots, leading to high computational expense.

The objective of this paper is to propose a fast and robust numerical algorithm that can be used to find the first intersection between a non-convex yield surface and an elastic trial stress path. This problem is further cast into a problem of finding the smallest positive root of a nonlinear function that has multiple roots.

2. M^2 Steffensen's method

The problem of finding the first intersection between a non-convex yield surface and an elastic trial stress path can be reformulated into a problem of finding the smallest positive root of a nonlinear function that has multiple roots:

$$f(x) = 0 \quad 0 \leqslant x \leqslant 1 \tag{2}$$

Furthermore, for elastoplastic problems, the initial stress state must be inside or on the initial yield surface, i.e. $f(0) \leqslant 0$. If the initial stress is on the yield surface, i.e. $f(0) = 0$, the root $x = 0$ is either the desired solution if the angle between the trial stress path and the norm of the yield surface is less than 90°, or otherwise irrelevant [12]. Hence we are only interested in the following situations:

$$f(0) < 0 \tag{3}$$