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A fast algorithm for solving banded Toeplitz systems

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ABSTRACT

A fast algorithm for solving systems of linear equations with banded Toeplitz matrices is presented. This new approach is based on extending the given matrix with several rows on the top and several columns on the right and to assign zeros and some nonzero constants in each of these rows and columns in such a way that the augmented matrix has a lower triangular Toeplitz structure. Stability of the algorithm is discussed and its performance is showed by numerical experiments.

1. Introduction

Toeplitz systems have been around for a long time and are encountered in various application fields. An interesting class of Toeplitz matrices consists of matrices having a banded Toeplitz form. In this paper, we consider the solution of the following n-by-n banded Toeplitz linear systems

$$Tx = f \tag{1}$$

where T is defined as

$$T = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-m_r} & & & \\ t_1 & t_0 & t_{-1} & \ddots & & & \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \\ t_{m_c} & & t_1 & t_0 & t_{-1} & & t_{-m_r} \\ & \ddots & & \ddots & \ddots & \ddots & \vdots \\ & & & & & \ddots & t_{-1} \\ & & & t_{m_c} & \cdots & t_1 & t_0 \end{pmatrix}, \tag{2}$$

x is the unknown vector, f is the right hand side, $t_{m_c} \neq 0$ and $t_{-m_r} \neq 0$.

Systems of linear equations with banded Toeplitz matrices as defined in (1) arise in a variety of applications in mathematics and engineering, see [1–3] and the references therein. In the literature there is a number of fast direct methods for large linear systems with banded Toeplitz matrices [4–11]. The most popular approach is the cyclic reduction invented by

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Bini and Meini [4,5] which seems to be the fastest and probably the most stable approach for the symmetric positive definite case or in the unsymmetric case with a suitable diagonal dominance. However, the cyclic reduction method sometimes fails to give accurate results in the unsymmetric case. Malyshev and Sadkane [10] recently proposed an alternative approach based on spectral factorization [12–14] and Woodbury's formula [15] which can give promising results in the symmetric and unsymmetric cases and when assuming that the band is not too large, i.e., $m = \max(m_c, m_r) \ll n$.

Although the algorithm described in [10] seems to be the method of choice, it presents some instability. In fact, when using a large band $m = \max(m_c, m_r)$ of T, the method proposed in [10] sometimes fails or requires a huge computational time to give a result. Therefore, we have introduced a new method for solving systems of linear equations with banded Toeplitz matrices (1) to resolve this problem.

Such an approach is based on extending the given matrix with several rows on the top and several columns on the right and to assign zeros and some nonzero constants in each of these rows and columns in such a way that the augmented matrix has a lower triangular Toeplitz structure. The computational cost of our algorithm is about $O(n \log n + m^2)$ flops.

The paper is organized as follows: Section 2 gives some definitions and theoretical results. An efficient algorithm for solving systems of linear equations with banded Toeplitz matrices is proposed in Section 3. We studied the error analysis in Section 4 and in Section 5 some numerical examples are given to put in evidence the potential advantages of our method with respect to the other well known methods, in terms of numerical stability and in terms of computational cost.

2. Some classical results

In this section, we shall present some definitions and properties of a Toeplitz matrix.

Definition 2.1. $T_n = \left[t_{ij}\right]_{i,j=0}^{n-1}$ is a Toeplitz matrix if $t_{ij} = t_{i+k,j+k}$ for all positive k (finite), that is, if all the entries of T_n are invariant in their shifts in the diagonal direction, so the matrix T_n is completely defined by its first row and its first column.

Toeplitz matrix of size n is completely specified by $2n - 1$ parameters, thus requiring less storage space than ordinary dense matrices. Moreover, many computations with Toeplitz matrices can be performed faster; this is the case, for instance, for the sum and the product by a scalar. Less trivial examples are given by the following results:

Proposition 2.1 (*[16]*). *The multiplication of a Toeplitz matrix of size n by a vector can be reduced to a multiplication of two polynomials of degree at most $2n$ and can be performed by the fast Fourier transform* [17] *in $O(n \log_2 n)$ arithmetic operations with a small constant.*

Proposition 2.2 (*[15]*).

1. *Let T_1 and T_2 be two lower (upper) triangular Toeplitz matrices. Then T_1T_2 is also a lower (upper) triangular Toeplitz matrix, and $T_1T_2 = T_2T_1$.*
2. *If T is a nonsingular lower (upper) triangular Toeplitz matrix, then T^{-1} is also a lower (upper) triangular Toeplitz matrix. Therefore, to obtain T^{-1}, we only require to compute the entries of its first column (row).*

Proposition 2.3 (*[18]*). *There are two types of fast direct algorithms for solving Toeplitz systems: Levinson-type algorithm which requires $O(n^2)$ flops, and Schur-type algorithm which requires $O(n \log^2 n)$ or $O(n \log^3 n)$.*

3. Main results

In the following, we give an algorithm which computes the solution of (1) by using the idea of embedding a banded Toeplitz matrix in a larger triangular Toeplitz matrix. Specifically, we embed T in a lower triangular Toeplitz matrix M of size $(n+p) \times (n+p)$ where the first column of M is given by

$$r = (t_{-m_r}, \ldots, t_{-1}, t_0, t_1, \ldots, t_{m_c}, \underbrace{0, \ldots, 0}_{n-(m_c+1)})^T.$$

More precisely,

$$M = \left(\begin{array}{ccccccc|ccc}
t_{-m_r} & & & & & & & & & \\
\vdots & \ddots & & & & & & & & \\
t_{-1} & \cdots & t_{-m_r} & & & & & & & \\
\hline
t_0 & t_{-1} & \cdots & t_{-m_r} & & & & & & \\
t_1 & t_0 & t_{-1} & & & & & & & \\
\vdots & \ddots & \ddots & \ddots & & \ddots & & & & \\
t_{m_c} & & t_1 & & t_{-1} & & t_{-m_r} & & & \\
 & \ddots & & \ddots & \ddots & \ddots & \vdots & t_{-m_r} & & \\
 & & & & & \ddots & t_{-1} & \vdots & \ddots & \\
 & & & t_{m_c} & \cdots & t_1 & t_0 & t_{-1} & \cdots & t_{-m_r}
\end{array}\right) \quad (3)$$