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A parallel fast algorithm for computing the Helmholtz integral operator in 3-D layered media

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ABSTRACT

In this paper, we propose a parallel fast algorithm for computing the product of the discretized Helmholtz integral operator in layered media and a vector in $O(N_qN_z^2N_xN_y\log N_xN_y)$ operations. Here $N_xN_yN_z$ is the number of sources and N_q is the number of quadrature points used in the evaluation of the Sommerfeld integral in the definition of layered media Green's function (for problems in thin-layer media, $N_z = O(1)$). Such a product forms the key step of many iterative solvers (such as the Krylov subspace based GMRES and BiCGSTAB) for linear systems arising from the integral equation methods for the Helmholtz equations. The fast solver is based on two important techniques which reduce the cost of quadrature summations in the Sommerfeld contour integral for Green's functions in 3-D layered media. The first technique is the removal of surface pole effects along the real axis integration contour by identifying the pole locations with a discrete wavelet transform; In the second technique, we apply a window-based high frequency filter to shorten the contour length. As a result, the integral operator for the 3-D layered media can be efficiently written as a sum of 2-D Hankel cylindrical integral operators, and the latter can be calculated by either a tree-code or a 2-D wideband fast multipole method in a fast manner. Numerical results show the efficiency and parallelism of the proposed fast algorithm.

1. Introduction

The electromagnetic simulation of scattering from objects embedded in a homogeneous or layered structure has many practical applications ranging from geoscience to quantum mechanics. Among the many numerical techniques, integral equation methods [19,25] have several advantages over the conventional finite difference method [32] or finite element method [30], such as the built-in radiation conditions at infinity, easy mesh generations for complex geometries, and a reduced number of unknowns. However, the main issue with integral equation methods in layered media, in contrast to the case of homogeneous media where fast solvers such as the fast multipole method (FMM) are available, is the lack of a fast solver for the dense matrix system arising from the discretized integral equation.

In general, to solve the matrix equation from the integral equation, a Krylov subspace based iteration method, such as GMRES [29] or BiCGSTAB [31], is used. A main step for the iterative solver involves the product of the matrix and a vector. A direct multiplication will cost $O(N^2)$ operations for a full $N \times N$ matrix. Many algorithms exist for a homogeneous background medium to reduce this cost to the order of $O(N)$ or $O(N\log N)$; among them are the FMM [27,28], the adaptive integral method (AIM) [3], and the FFT method [4]. However, the speed-up of integral equation methods for layered media still faces

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much challenges. There have been several attempts to construct fast integral methods for layered media [20]. The natural way is to formulate the scattering problems using Green's functions of the layered media. As the layered media Green's function is expressed as a Hankel transform, defined in terms of the Sommerfeld integral, its calculation is time consuming and also dictates how the Helmholtz integral operator can be implemented. Many methods have been proposed to address the difficulties of the Sommerfeld integral, for example, the complex image method (CIM) via Prony's approximation [2,12], the fast Hankel transform [22], and the steepest decent path for the Sommerfeld integration [13,14,24]. In this paper, techniques such as high frequency filtering and an adaptive quadrature formula (with surface pole identification by a discrete wavelet transform) will be used to minimize the cost of computing the Helmholtz integral operator. The key component of the fast algorithm relies on decomposing the integral operator for the 3-D layered media as a series of operators defined by cylindrical waves (Hankel kernels) within a range of wave numbers resulting from the quadrature formula for the Sommerfeld integral. The cylindrical wave integral operator will be then implemented by a local expansion tree-code for the Bessel function and a wideband FMM (wFMM) [9–11,15], and the latter can be downloaded from http://fastmultipole.org/ [16].

This paper will be organized as follows: in Section 2, the spectral form of the Green's function will be derived and validated with known analytical solutions. Section 3 starts with the Green's function in physical space using the Hankel transform, and then techniques mentioned above are presented to minimize the quadrature cost in the contour integral for the Hankel transform. In Section 4, we will present the fast algorithm for implementing the Helmholtz integral operator in 3-D layered media with the local expansion tree-code and wFMM, and numerical performance of the proposed fast algorithms for thin-layer structures will also be presented. Conclusions are given in Section 5.

2. Spectral form for Green's functions in layered media

The 3-D scalar Helmholtz equation with a point source at $\mathbf{r}' = (0,0,z')$ in an N-layer structure (Fig. 1) with layer locations $d_i,\ 0 \leqslant i \leqslant N-1$, is defined by the following equation,

$$\nabla \frac{1}{m_i} \nabla G_i(\mathbf{r},\mathbf{r}') + k_i^2 G_i(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r},\mathbf{r}'), \quad i = 0,1,\ldots,N, \tag{1}$$

where i indicates the ith layer. By taking the Fourier transform in the horizontal plane to transform x and y to spectral variables k_x and k_y, respectively, we obtain

$$\frac{d}{dz}\frac{1}{m_i}\frac{d}{dz}\widehat{G}_i(k_x,k_y,z;z') - (k_\rho^2 - k_i^2)\widehat{G}_i(k_x,k_y,z;z') = -\frac{1}{2\pi}\delta(z-z'), \tag{2}$$

where

$$k_\rho^2 = k_x^2 + k_y^2. \tag{3}$$

Due to cylindrical symmetry in the (k_x,k_y) variables, Eq. (2) simplifies to

$$\frac{d}{dz}\frac{1}{m_i}\frac{d}{dz}\widehat{G}_i(k_\rho,z;z') - u_i^2\widehat{G}_i(k_\rho,z;z') = -\frac{1}{2\pi}\delta(z-z'), \tag{4}$$

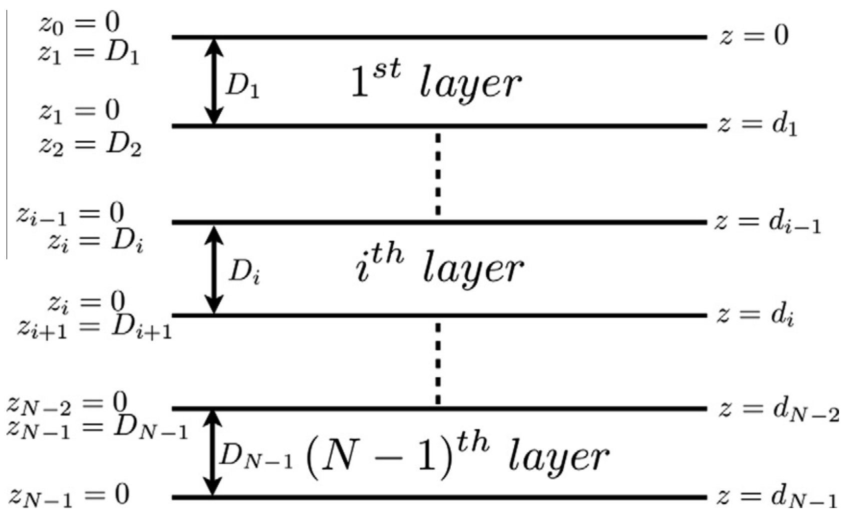

Fig. 1. Layered structure with a local coordinate system, $z_i = z - d_i$.