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A faster algorithm for packing branchings in digraphs

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ABSTRACT

We consider the problem of finding an integral (and fractional) packing of branchings in a capacitated digraph with root-set demands. Schrijver described an algorithm that returns a packing with at most $m + n^3 + r$ branchings that makes at most $m(m + n^3 + r)$ calls to an oracle that basically computes a minimum cut, where n is the number of vertices, m is the number of arcs and r is the number of root-sets of the input digraph. Leston-Rey and Wakabayashi described an algorithm that returns a packing with at most $m + r - 1$ branchings but makes a large number of oracle calls. In this work we provide an algorithm, inspired on ideas of Schrijver and in a paper of Gabow and Manu, that returns a packing with at most $m+r-1$ branchings and makes at most $(m+r+2)n$ oracle calls. Moreover, for the arborescence packing problem our algorithm provides a packing with at most $m-n+2$ arborescences – thus improving the bound of m of Leston-Rey and Wakabayashi – and makes at most $(m-n+5)n$ oracle calls.

1. Introduction

Integral packing problems form an important class of combinatorial optimization problems. A general algebraic version of a packing problem can be formalized as follows. Let $\mathcal{B}$ be a set of vectors in an m-dimensional vector space. Suppose we are given a non-negative integral vector c and we are asked to find (if possible) positive integers $y_1, \ldots, y_k$ and vectors $b_1, \ldots, b_k \in \mathcal{B}$ such that

$$y_1 b_1 + \cdots + y_k b_k \leqslant c.$$

When we require equality above, we are asking whether c belongs to the integer cone generated by the vectors in $\mathcal{B}$. In combinatorial applications, the set $\mathcal{B}$ is typically much larger than m. For example, $\mathcal{B}$ could be the set of (incidence vectors of) bases of a matroid whose ground set has m elements or could be the set of r-arborescences of a digraph with m arcs. From the perspective of an algorithmic designer, it is essential to find a packing with a polynomial number of members of $\mathcal{B}$. Moreover, from a theoretical point of view it would be very interesting to find the smallest upperbound k for which it is always possible to pack at most k elements of $\mathcal{B}$ independently of the choice of c. This latter problem is related to the study of Hilbert bases introduced by Cook, Fonlupt and Schrijver [2].

In this paper we are concerned with the case in which $\mathcal{B}$ corresponds to the set of branchings (with many root-sets) of digraphs. We note that the problem we consider here does not fit exactly in the general setting we have defined.

In a seminal paper Edmonds [4] characterized when a capacitated digraph with root-set demands has an integral packing of branchings. Lovász [12] established the fundamental ideas and proof techniques for problems involving packing

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of arborescences and branchings [1,6,7,10]. In this paper we investigate this problem from the point of view of finding efficiently a packing with few branchings.

Schrijver [15] described an algorithm that returns a packing with at most $m + n^3 + r$ distinct branchings that makes at most $m(m + n^3 + r)$ calls to an oracle that basically computes a minimum cut, where n is the number of vertices, m is the number of arcs, and r is the number of root-sets of the input digraph. Pevzner [14] considered the special case of finding a maximum integral packing of r-arborescences in a capacitated digraph and proved there exists such a packing with $O(nm)$ distinct arborescences. Gabow and Manu [8], also for this special case, provided an algorithm that returns a packing with at most $m + n - 2$ distinct arborescences whose time complexity is $O(\min\{n, \lg C\}n^2 m \lg(n^2/m))$, where C is the largest capacity of an arc. They also showed an upperbound of m for the number of distinct arborescences in a fractional packing. One of the key ideas in their algorithm is to keep a laminar family of cuts in order to bound the number of arborescences used in the final packing. Leston-Rey and Wakabayashi [11] described an algorithm for a general framework that implies an algorithm that returns an integral packing of branchings using at most $m+r-1$ distinct branchings, which in turn implies that there exists an integral packing of arborescences using at most m distinct arborescences. Though their algorithm is oracle-polynomial time and improves on the best known upper bounds for the packing size, it also requires a large number of oracle calls.

In this work we provide an algorithm for packing branchings for both the fractional and the integral version. The algorithm returns a packing with at most $m + r - 1$ branchings and makes at most $(m + r + 2)n$ oracle calls.

This paper is organized as follows. In the remaining of this section we introduce some basic notation, recall Edmonds' theorem and present our main result (Theorem 2) and its consequences. In Section 2, we discuss some concepts and auxiliary results which we use throughout the paper. In Section 3 we present our main contribution: a new algorithm for packing branchings in a network. Finally, in Section 4 we describe the pre-processing step of our algorithm — this is required to ensure that the packing produced by the algorithm is "small".

Preliminaries Before we begin, let us state a few preliminary definitions. For a function $f : X \to \mathbb{R}_+$, and $Y \subseteq X$, we write $f(Y)$ to denote the sum $\sum[f(y) \mid y \in Y]$. The *support* of f, denoted by f^+, is the set $\{x \in X \mid f(x) > 0\}$. Let B be a subset of some set E. The *characteristic function* of B is the function $\chi^B : E \to \{0, 1\}$ defined by

$$\chi^B(e) := \begin{cases} 1 & \text{if } e \in B, \\ 0 & \text{otherwise} \end{cases}$$

for each $e \in E$. We use characteristic functions without explicitly stating its domain, since the context will imply to which domain we refer. A function is *integral* if its image is a subset of the set $\mathbb{N}$ of nonnegative integers.

For a set V, a subset $\mathcal{P}$ of 2^V, we write $\bigcup \mathcal{P}$ to denote the set $\{u \mid u \in U \text{ for some } U \in \mathcal{P}\}$.

For a property P, we write

$$[P] := \begin{cases} 1, & \text{if } P \text{ is true} \\ 0, & \text{otherwise.} \end{cases}$$

A *digraph* is a pair $D = (V, A)$, where V is a finite set of *vertices*, and A is a finite set of *arcs*. Each arc is associated with two vertices, its *ends*, which are called its *tail* and its *head*. Let S be a subset of V. We write $\bar{S}$ to denote $V \setminus S$. We say that an arc $a \in A$ *enters* S if the tail of a is in $\bar{S}$ and the head of a is in S, otherwise we say that *a avoids* S. We write $\rho(S)$, or simply ρS, to denote the set of arcs of D that enter S. Finally, set

$$\mathcal{C}_S := \{\emptyset \neq U \subseteq V \mid U \cap S = \emptyset\}.$$

An S-*cover* is a subset B of arcs such that $B \cap \rho U \neq \emptyset$ for each $U \in \mathcal{C}_S$. An S-*branching* is a minimal S-cover. For most of the proofs in this paper we only require that B is an S-cover rather than an S-branching. We also say that $B \subseteq A$ *avoids* $\mathcal{P} \subseteq 2^V$ if $B \cap \rho U = \emptyset$ for each $U \in \mathcal{P}$. When $B = \{a\}$ we just say that *a avoids* $\mathcal{P}$.

Let $D = (V, A)$ be a digraph, $c : A \to \mathbb{R}_+$ an *arc capacity function* and $\mu : 2^V \setminus \{\emptyset\} \to \mathbb{R}_+$ a *demand function*. The triple $\mathbb{F} = (D, c, \mu)$ is called a *network*. Each set in μ^+ is called a *root-set*. It is convenient to set

$$c(U) := \sum[c(a) \mid a \in \rho U], \text{ and}$$
$$c(U, W) := \sum[c(a) \mid a \in A \text{ has one end in } U \text{ and the other in } W]$$

for each $U, W \subseteq V$. It is well-known that

$$c(U) + c(W) = c(U \cup W) + c(U \cap W) + c(U \setminus W, W \setminus U), \tag{1}$$

for each $U, W \subseteq V$. In other words, c is a *submodular* function.

For each nonempty $U \subseteq V$, we define the *demand induced* by μ by setting

$$p(U) := \sum[\mu(R) \mid R \in \mu^+, U \in \mathcal{C}_R].$$

It is straightforward to verify that p is a *supermodular* function, that is,

$$p(U) + p(W) \leqslant p(U \cup W) + p(U \cap W), \tag{2}$$