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Fast algorithms for spherical harmonic expansions, II

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Abstract

We provide an efficient algorithm for calculating, at appropriately chosen points on the two-dimensional surface of the unit sphere in $\mathbb{R}^3$, the values of functions that are specified by their spherical harmonic expansions (a procedure known as the inverse spherical harmonic transform). We also provide an efficient algorithm for calculating the coefficients in the spherical harmonic expansions of functions that are specified by their values at these appropriately chosen points (a procedure known as the forward spherical harmonic transform). The algorithms are numerically stable, and, if the number of points in our standard tensor-product discretization of the surface of the sphere is proportional to l^2, then the algorithms have costs proportional to $l^2 \ln(l)$ at any fixed precision of computations. Several numerical examples illustrate the performance of the algorithms.

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1. Introduction

Over the past several decades, the fast Fourier transform (FFT) and its variants (see, for example [15]) have had an enormous impact across the sciences. The FFT is an efficient algorithm for computing, for any positive integer n and complex numbers $\beta_0, \beta_1, \ldots, \beta_{n-2}, \beta_{n-1}$, the complex numbers $\alpha_0, \alpha_1, \ldots, \alpha_{n-2}, \alpha_{n-1}$ defined by

$$\alpha_j = \sum_{k=0}^{n-1} \beta_k \, e_k(x_j) \tag{1}$$

for $j = 0, 1, \ldots, n-2, n-1$, where $e_0, e_1, \ldots, e_{n-2}, e_{n-1}$ are the functions defined on $[-1, 1]$ by

$$e_k(x) = \exp\left(\frac{\pi \mathrm{i}(2k-n)x}{2}\right) \tag{2}$$

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for $k = 0, 1, \ldots, n-2, n-1$, and $x_0, x_1, \ldots, x_{n-2}, x_{n-1}$ are the real numbers defined by

$$x_k = \frac{2k - n}{n} \tag{3}$$

for $k = 0, 1, \ldots, n-2, n-1$. The FFT is efficient in the sense that there exists a reasonably small positive real number C such that, for any positive integer $n \geqslant 10$, the FFT requires at most $C\,n\ln(n)$ floating-point operations to compute $\alpha_0, \alpha_1, \ldots, \alpha_{n-2}, \alpha_{n-1}$ in (1) from $\beta_0, \beta_1, \ldots, \beta_{n-2}, \beta_{n-1}$. In contrast, evaluating the sum in (1) separately for every $j = 0, 1, \ldots, n-2, n-1$ costs at least n^2 operations in total.

It is desirable to have an analogue of the FFT for functions defined on the two-dimensional surface of the unit sphere in $\mathbb{R}^3$, in the following sense. The spherical harmonic expansion of a bandlimited function f on the surface of the sphere has the form

$$f(\theta, \varphi) = \sum_{k=0}^{2l-1} \sum_{m=-k}^{k} \beta_k^m \overline{P}_k^{|m|}(\cos(\theta)) e^{im\varphi}, \tag{4}$$

where (θ, φ) are the standard spherical coordinates on the two-dimensional surface of the unit sphere in $\mathbb{R}^3, \theta \in (0, \pi)$ and $\varphi \in (0, 2\pi)$, and $\overline{P}_k^{|m|}$ is the normalized associated Legendre function of degree k and order $|m|$, defined on $(-1, 1)$ via the formula

$$\overline{P}_k^{|m|}(x) = \sqrt{\frac{2k+1}{2} \frac{(k-|m|)!}{(k+|m|)!}} \sqrt{1-x^2}^{|m|} \frac{d^{|m|}}{dx^{|m|}} P_k(x), \tag{5}$$

where P_k is the Legendre polynomial of degree k (see, for example, chapter 8 of [1]). (Please note that the superscript m in β_k^m denotes an index, rather than a power.) Obviously, the expansion (4) contains $4l^2$ terms. The complexity of the function f determines l.

In many areas of scientific computing, particularly those using spectral methods for the numerical solution of partial differential equations, we need to evaluate the coefficients β_k^m in an expansion of the form (4) for a function f given by a table of its values at a collection of appropriately chosen nodes on the two-dimensional surface of the unit sphere. Conversely, given the coefficients β_k^m in (4), we often need to evaluate f at a collection of points on the surface of the sphere. The former is known as the forward spherical harmonic transform, and the latter is known as the inverse spherical harmonic transform. A standard discretization of the surface of the sphere is the "tensor-product," consisting of all pairs of the form (θ_k, φ_j), with $\cos(\theta_0), \cos(\theta_1), \ldots, \cos(\theta_{2l-2}), \cos(\theta_{2l-1})$ being the Gauss–Legendre quadrature nodes of degree $2l$, that is,

$$-1 < \cos(\theta_0) < \cos(\theta_1) < \cdots < \cos(\theta_{2l-2}) < \cos(\theta_{2l-1}) < 1 \tag{6}$$

and

$$\overline{P}_{2l}^0(\cos(\theta_k)) = 0 \tag{7}$$

for $k = 0, 1, \ldots, 2l-2, 2l-1$, and with $\varphi_0, \varphi_1, \ldots, \varphi_{4l-3}, \varphi_{4l-2}$ being equispaced on the interval $(0, 2\pi)$, that is,

$$\varphi_j = \frac{2\pi\left(j + \frac{1}{2}\right)}{4l - 1} \tag{8}$$

for $j = 0, 1, \ldots, 4l-3, 4l-2$. This leads immediately to numerical schemes for both the forward and inverse spherical harmonic transforms with costs proportional to l^3.

Indeed, given a function f defined on the two-dimensional surface of the unit sphere by (4), we can rewrite (4) in the form

$$f(\theta, \varphi) = \sum_{m=-2l+1}^{2l-1} e^{im\varphi} \sum_{k=|m|}^{2l-1} \beta_k^m \overline{P}_k^{|m|}(\cos(\theta)). \tag{9}$$

For a fixed value of θ, each of the sums over k in (9) contains no more than $2l$ terms, and there are $4l - 1$ such sums (one for each value of m); since the inverse spherical harmonic transform involves $2l$ values $\theta_0, \theta_1, \ldots, \theta_{2l-2}, \theta_{2l-1}$, the cost of evaluating all sums over k in (9) is proportional to l^3. Once all sums over k have been evaluated, each sum over m may be evaluated for a cost proportional to l (since each of them