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An effective heuristic algorithm for sum coloring of graphs

Qinghua Wu, Jin-Kao Hao *

LERIA, Université d'Angers 2 Boulevard Lavoisier, 49045 Angers Cedex 01, France

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ABSTRACT

Given an undirected graph $G=(V,E)$, the minimum sum coloring problem (MSCP) is to find a legal vertex coloring of G, using colors represented by natural numbers $(1,2,\ldots)$ such that the total sum of the colors assigned to the vertices is minimized. In this paper, we present EXSCOL, a heuristic algorithm based on independent set extraction for this NP-hard problem. EXSCOL identifies iteratively collections of disjoint independent sets of equal size and assign to each independent set the smallest available color. For the purpose of computing large independent sets, EXSCOL employs a tabu search based heuristic. Experimental evaluations on a collection of 52 DIMACS and COLOR2 benchmark graphs show that the proposed approach achieves highly competitive results. For more than half of the graphs used in the literature, our approach improves the current best known upper bounds.

1. Introduction

Given an undirected graph $G=(V,E)$ with vertex set V and edge set E, an independent set I of G is a subset of V such that no two vertices in I are joined by an edge in E. A legal k-coloring of G is a partition of V into k disjoint independent sets $I_1,I_2,\ldots,I_k$. The smallest integer k such that a legal k-coloring exists for G is the *chromatic number* of G (denoted by $\chi(G)$). The well-known NP-hard *graph vertex coloring* problem is to find the chromatic number of a graph [7]. In this paper, we are interested in a related problem known as the *minimum sum coloring problem* (MSCP for short).

The MSCP is to find a vertex coloring $c=\{I_1,\ldots,I_k\}$ of G such that the following total sum of the colors is minimized:

$$Sum(c)=\sum_{i=1}^{k}\sum_{v\in I_i} i. \quad (1)$$

The optimal (smallest) value of this sum is called the *chromatic sum* of G and denoted by $\sum(G)$. The number k of the k-coloring leading to the *chromatic sum* is called the strength of the graph and denoted by $s(G)$. It is clear that $s(G)$ is lower bounded by $\chi(G)$, i.e. $s(G)\geq\chi(G)$.

The minimum sum coloring problem is known to be NP-hard in the general case [14]. In addition to its theoretical significance as a difficult combinatorial problem, the MSCP is notable for its ability to formulate a number of important problems, including those from VLSI design, scheduling and resource allocation [1,19].

During the past two decades, the MSCP has been studied essentially from a theoretical point of view and special cases (e.g. tree, interval graphs, line graphs, etc.) have been identified which admit efficient approximation algorithms or polynomial algorithms [1–3,8,9,12,19,23]. For the purpose of practical solving of the general MSCP, several heuristic algorithms have recently been proposed to find suboptimal solutions. Notice that this heuristic based approach is expected to find good approximate solutions within reasonable computing time, but without provable solution quality.

For instance, Kokosiński and Kawarciany proposed a parallel genetic algorithm [13]. In [17], Y. Li et al. presented MRLF, an effective greedy algorithm based on the well-known RLF graph coloring heuristic [16]. Moukrim et al. showed a technique for computing the lower bound for the MSCP based on extraction of specific partial graphs [21]. Bouziri and Jouini adapted a tabu coloring algorithm to sum coloring [4]. Douiri and Elbernoussi illustrated a hybrid algorithm which combines a genetic algorithm with a local search heuristic [5]. Finally, in [1] Bar-Noy et al. presented a theoretical study of a heuristic algorithm based on finding iteratively maximum independent sets (MaxIS) and showed that the MaxIS is a 4-approximation to the MCSP, which is a tight bound to within a factor of 2. Nevertheless, the practical performance of this heuristic was not verified with computational experiments.

In this paper, we present a heuristic algorithm called EXSCOL for the MSCP based on the idea of independent set extraction. The similar approach was initially applied to the vertex coloring problem [25]. Basically, EXSCOL iteratively extracts from the graph as many large disjoint independent sets of equal size as possible. For each extracted independent set, we assign to it the smallest available color (colors are represented by natural

* Corresponding author.
E-mail addresses: wu@info.univ-angers.fr (Q. Wu), Jin-Kao.Hao@univ-angers.fr, hao@info.univ-angers.fr (J.-K. Hao).

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numbers 1, 2 ...). This process is repeated until the graph becomes empty. The rationale behind this approach is that by extracting many large disjoint independent sets, we naturally favor the construction of large color classes and reduce the number of needed color classes, leading to a reduced total sum of colors. Since computing a maximum independent set of a graph is NP-hard [7], we employ the tabu search based heuristic introduced in [26] to find large independent sets.

We present experimental results on a set of 52 benchmark graphs in the literature, showing that the proposed algorithm achieves very competitive results with respect to the existing sum coloring heuristics. Indeed, for more than half of the instances used in the literature, the proposed approach improves the current best known results. Furthermore, we assess the relative performance of two other solution methods using respectively the conventional independent set extraction strategy and graph vertex coloring algorithms.

The rest of this paper is organized as follows. In the next section we give a formal description of the proposed EXSCOL algorithm. In Section 3, computational results are presented and compared with several state-of-the-art algorithms from the literature. In Section 4, we show additional studies and comparisons with respect to two other solutions methods, followed by conclusions in Section 5.

2. EXSCOL: an algorithm for the MSCP

2.1. Rationale and general procedure

Let $c=\{I_1,\ldots,I_k\}$ be a legal coloring of graph $G=(V,E)$, each independent set I_i is a color class of c such that all the vertices $v \in I_i$ receive color i. Given the coloring c, its sum of colors $Sum(c)$ according to Eq. (1) counts the total sum of the colors induced by c. Suppose $|I_1| \geq |I_2| \geq \cdots \geq |I_1|$, Eq. (1) can be rewritten as follows:

$$Sum(c) = 1 \cdot |I_1| + 2 \cdot |I_2| + \cdots + k \cdot |I_k| = \sum_{i=1}^{k} i \cdot |I_i|. \tag{2}$$

It is clear that the sum depends on both the number k of the used colors and the size of the color classes. To minimize this sum, one can try to construct large color classes and assign to them small colors. For this purpose, one can remove iteratively the maximum number of disjoint independent sets of the maximum size from the graph until the graph becomes empty. However, both computing a maximum independent set of a graph and a maximum set of disjoint sets (which is the maximum set packing problem) are NP-hard problems [7]. Consequently, heuristics are needed in the general case to find approximate solutions, in particular for large graphs.

The proposed EXSCOL algorithms follows this basic idea and can be summarized by the following procedure:

(1) identify an independent set of the largest size possible from the graph;
(2) identify as many pairwise disjoint independent sets of that size as possible and extract them from the graph;
(3) assign to each extracted independent set the smallest available color (the first color used is 1);
(4) stop if the graph becomes empty, goto Step 1 otherwise.

Fig. 1 illustrates how this approach works on a graph with nine vertices. At the first step, we find a maximum independent set of size 3 (e.g. $\{A,D,H\}$). Then we try to identify as many disjoint independent sets of size 3 as possible from the graph, leading to three disjoint independent sets $\{A,B,F\},\{C,E,H\},\{D,G,I\}$. We assign colors 1, 2, 3 to these independent sets. Since the graph becomes empty after removing these independent sets, the procedure stops. We obtain a coloring $c=\{\{A,B,F\},\{C,E,H\},\{D,G,I\}\}$ with $Sum(c)=18$ for the graph.

2.2. The EXSCOL algorithm

The proposed EXSCOL algorithm (Algorithm 1) implements the general procedure given in Section 2.1. EXSCOL starts by identifying a first largest possible independent set I_M (line 4) whose size $|I_M|$ is used later to build a pool M of independent sets of that size (lines 5–15). The search for a new independent set of size $|I_M|$ stops when the number of independent sets contained in M reaches a desired threshold (M_{max}) or when no new independent set of that size is found after p_{max} consecutive tries.

From M, EXSCOL tries to determine a maximum number of disjoint independent sets (line 16). This task corresponds in fact to the maximum set packing problem, which is equivalent to the maximum clique (thus the maximum independent set) problem [7]. Given $M=\{I_1,\ldots,I_n\}$, we construct an instance of the independent set problem as follows.

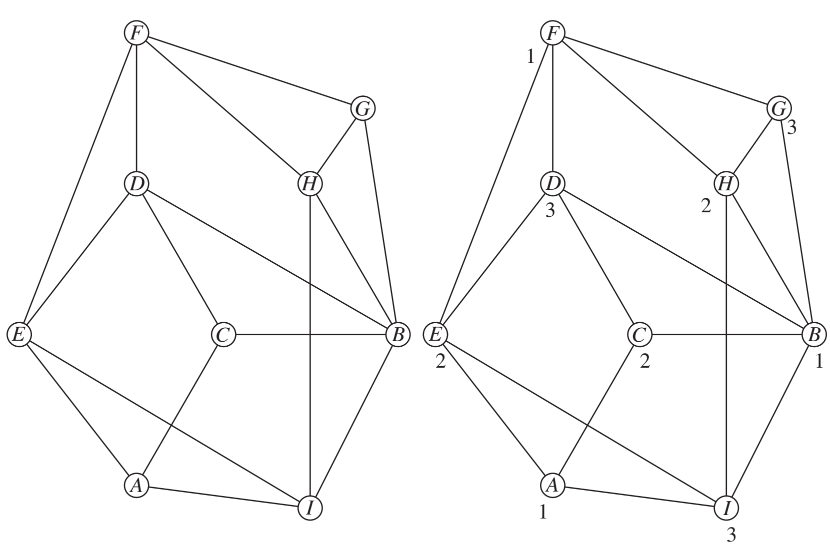

Fig. 1. An illustration of the proposed EXSCOL algorithm.