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Lower bounds and heuristic algorithms for the k_i-partitioning problem

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Abstract

We consider the problem of partitioning a set of positive integers values into a given number of subsets, each having an associated cardinality limit, so that the maximum sum of values in a subset is minimized, and the number of values in each subset does not exceed the corresponding limit. The problem is related to scheduling and bin packing problems. We give combinatorial lower bounds, reduction criteria, constructive heuristics, a scatter search approach, and a lower bound based on column generation. The outcome of extensive computational experiments is presented.

Keywords: Partitioning; Cardinality constraints; Scheduling; Parallel machines; Bin packing; Scatter search; Column generation

1. Introduction

Given *n items* I_j $(j = 1, \ldots, n)$, each characterized by an integer positive *weight* w_j, and m positive integers k_i $(i = 1, \ldots, m)$ with $m < n \leqslant \sum_{i=1}^{m} k_i$, the k_i-*Partitioning Problem* (k_i-PP) is to partition the items into m *subsets* $S_1, \ldots, S_m$ so that $|S_i| \leqslant k_i$ $(i = 1, \ldots, m)$ and the maximum total weight of a subset is a minimum. The problem was introduced by Babel et al. [1] and finds possible applications, e.g., in Flexible Manufacturing Systems. Assume that we have to execute a set of operations of n different types, and that the operations of type j, requiring in total a time w_j, must be assigned to the same cell: If the capacity of the specific tool magazine of each cell imposes a limit on the number of types of operation the cell can perform, then k_i-PP models the problem of completing the process in minimum total time.

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A famous scheduling problem (usually denoted as $P\|C_{\max}$) asks for assigning *n jobs*, each having an integer positive *processing time* w_j, to m identical parallel *machines* $M_i\,(i = 1,\ldots,m)$, each of which can process at most one job at a time, so as to minimize their total completion time (*makespan*). By associating items to jobs and subsets to machines, it is clear that k_i-PP is the generalization of $P\|C_{\max}$ arising when an additional constraint imposes an upper bound k_i on the number of jobs that can be processed by machine M_i. Since $P\|C_{\max}$ is known to be strongly NP-hard, the same holds for k_i-PP.

Another special case of k_i-PP, that also generalizes $P\|C_{\max}$, is the $P|\# \leqslant k|C_{\max}$ scheduling problem, in which an identical limit k is imposed on the maximum number of jobs that can be assigned to any machine. Upper and lower bounds for this problem have been developed by Babel et al. [1], Dell'Amico and Martello [6] and Dell'Amico et al. [4].

The *Bin Packing Problem (BPP)* too is related to k_i-PP. Here we are given n items, each having an associated integer positive *weight* w_j, and an unlimited number of identical containers (*bins*) of *capacity* c: The problem is to assign all items to the minimum number of bins so that the total weight in each bin does not exceed the capacity. Problem BPP can be seen as a "dual" of $P\|C_{\max}$: By determining the minimum c value for which an m-bin BPP solution exists, we also solve the corresponding $P\|C_{\max}$ problem. By introducing a limit k on the number of items that can be assigned to any bin, we similarly obtain a dual of $P|\# \leqslant k|C_{\max}$. In order to obtain a dual of k_i-PP, we can impose the given limits $k_i\,(i = 1,\ldots,m)$ to the first m bins, and a limit equal to one to all other bins.

The dual relations above have been used to obtain heuristic algorithms and lower bounds for $P\|C_{\max}$ (Coffman et al. [2], Hochbaum and Shmoys [16], Dell'Amico and Martello [5]) and $P|\# \leqslant k|C_{\max}$ (Dell'Amico and Martello [6]).

In this paper we study upper and lower bounds for k_i-PP, either obtained by generalizing algorithms from the literature so as to handle the cardinality constraints, or originally developed for the considered problem. In Section 2 we present lower bounds and reduction criteria. In Section 3 we examine generalizations of heuristic algorithms and of a scatter search approach. In Section 4 we propose a lower bound based on a column generation approach, that makes use of the above mentioned relations with BPP. The effectiveness of the proposed approaches is computationally analyzed in Section 5 through extensive computational experiments on randomly generated data sets.

Without loss of generality, we will assume in the following that items I_j are sorted by non-increasing w_j value, and subsets S_i by non-decreasing k_i value.

2. Lower bounds and reduction criteria

By introducing binary variables $x_{ij}\,(i = 1,\ldots,m; j = 1,\ldots,n)$ taking the value 1 iff item I_j is assigned to subset S_i, an ILP model of k_i-PP can be written as

$$\min z \tag{1}$$

$$\sum_{j=1}^{n} w_j x_{ij} \leqslant z \quad (i = 1,\ldots,m), \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} = 1 \quad (j = 1,\ldots,n), \tag{3}$$

$$\sum_{j=1}^{n} x_{ij} \leqslant k_i \quad (i = 1,\ldots,m), \tag{4}$$

$$x_{ij} \in \{0,1\} \quad (i = 1,\ldots,m;\ j = 1,\ldots,n), \tag{5}$$

where variable z represents the maximum weight of a subset.