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A heuristic algorithm for computing the max–min inverse fuzzy relation

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Abstract

The paper addresses a classical problem of computing approximate max–min inverse fuzzy relation. It is an NP-complete problem for which no polynomial time algorithm is known till this date. The paper employs a heuristic function to reduce the search space for finding the solution of the problem. The time-complexity of the proposed algorithm is $O(n^3)$, compared to $O(k^n)$, which is required for an exhaustive search in the real space of $[0, 1]$ at k regular intervals of interval length $(1/k)$.

Keywords: Algorithms; Max–min inverse fuzzy relation; Heuristics

1. Introduction

Let X and $Y \subseteq$ r be two universal sets. A fuzzy relation that describes a mapping from X to Y $(X \to Y)$ generally is a fuzzy subset of $X \times Y$, where '$\times$' denotes a cartesian product [18]. Formally, a fuzzy relation $\mathbf{R}$ is defined by

$$\mathbf{R}(x, y) = \{((x, y), \mu_{\mathbf{R}}(x, y)) \,|\, (x, y) \in X \times Y\}, \tag{1}$$

where $\mu_{\mathbf{R}}(x, y)$ refers to the membership of (x, y) to belong to the fuzzy relation $\mathbf{R}(x, y)$. Fuzzy 'composition' [8] is an operation, by which fuzzy relations in different product space can be combined with each other. There exist different

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versions of 'composition'. The 'max–min' composition, which is most popular among them is defined below. Let $X, Y, Z \subseteq$ r be three universal sets and $R_1(x,y), (x,y) \in X \times Y$ and $R_2(y,z), (y,z) \in Y \times Z$ be two fuzzy relations. The max–min composition of $\mathbf{R}_1$ and $\mathbf{R}_2$, denoted by $\mathbf{R}_1 \circ \mathbf{R}_2$ is then a fuzzy set, is defined by

$$\mathbf{R}_1 \circ \mathbf{R}_2 = \left\{ (x,z), \max_y \{ \min\{ \mu_{\mathbf{R}_1}(x,y), \mu_{\mathbf{R}_2}(y,z) \} \} \right\}, \tag{2}$$

where $x \in X,\ y \in Y$ and $z \in Z$. For brevity, we shall use '$\wedge$' and '$\vee$ ' to denote 'min' and 'max' operators, respectively. Thus expression (2) can be re-written as

$$\mathbf{R}_1 \circ \mathbf{R}_2 = \left\{ (x,z), \bigvee_y \{ \mu_{\mathbf{R}_1}(x,y) \wedge \mu_{\mathbf{R}_2}(y,z) \} \right\}. \tag{3}$$

We use $\mu_{\mathbf{R}_1 \circ \mathbf{R}_2}(x,z)$ to denote the membership function of (x,z) in the max–min composition relation $\mathbf{R}_1 \circ \mathbf{R}_2$ is defined by

$$\mu_{\mathbf{R}_1 \circ \mathbf{R}_2}(x,z) = \bigvee_y \{ \mu_{\mathbf{R}_1}(x,y) \wedge \mu_{\mathbf{R}_2}(y,z) \} \tag{4}$$

1.1. Fuzzy max–min inverse relations

Let $X = \{x_1, x_2, \ldots, x_n\},\ Y = \{y_1, y_2, \ldots, y_m\}$ and $Z = \{z_1, z_2, \ldots, z_l\}$ be three universal sets and $\mathbf{R}_1, \mathbf{R}_2$ be two fuzzy relations on $X \times Y$ and $Y \times Z$, respectively. Again, let $\mathbf{R}_1 \circ \mathbf{R}_2 = \mathbf{I}$, where $\mathbf{I}$ denotes an identity relation, such that $\mu_{\mathbf{R}_1 \circ \mathbf{R}_2}(x,z) = \mathbf{I}$, when $x = x_i \in X$ and $z = z_i \in Z$ and $\mu_{\mathbf{R}_1 \circ \mathbf{R}_2}(x,z) = 0$, otherwise. Under this circumstances, we call R_1, *the max–min pre-inverse relation to* $\mathbf{R}_2$ and $\mathbf{R}_2$, *the max–min post-inverse relation to* $\mathbf{R}_1$. Unfortunately, $\mathbf{R}_1 \circ \mathbf{R}_2 = \mathbf{I}$ is true, only when $\mathbf{R}_1 = \mathbf{R}_2 = \mathbf{I}$. We thus define $\mathbf{R}_1$ as the *approximate max–min pre-inverse relation to* $\mathbf{R}_2$, when $\mathbf{R}_1 \circ \mathbf{R}_2 = \mathbf{I}'$, such that $\mathbf{I}'$ is sufficiently close to $\mathbf{I}$ with respect to a Euclidean norm of the difference $(\mathbf{I} - \mathbf{I}')$, estimated by

$$D = \left[\sum_{\forall z} \sum_{\forall x} \{ \mu_{\mathrm{I}}(x,z) - \mu_{\mathrm{I}'}(x,z) \}^2 \right]^{1/2},$$

where D should not exceed a small pre-defined real number. The definition of *approximate post-inverse relation* to $\mathbf{R}_1$ may also be given analogously.

1.2. Best approximate pre-inverse relation

Let $\mathbf{Q}$ be a set of fuzzy relations of $\mathbf{R}_1$, such that for all $\mathbf{R}_1 \in \mathbf{Q}$, there exists an $\mathbf{R}_2$ with $\mathbf{R}_1 \circ \mathbf{R}_2 = \mathbf{I}'$ and