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Generic determinacy of Nash equilibrium in network-formation games ☆

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ABSTRACT

This paper proves the generic determinacy of Nash equilibrium in network-formation games: for a generic assignment of utilities to networks, the set of probability distributions on networks induced by Nash equilibria is finite.

1. Introduction

A basic desideratum when applying noncooperative game theory is to have a finite set of probability distributions on outcomes derived from equilibria.[1] When utilities are defined over the relevant outcome space, it is well known that this is generically the case when we can assign a different outcome to each pure strategy profile (Harsanyi, 1973), or to each ending node of an extensive-form game (Kreps and Wilson, 1982).[2]

A *game-form* endows players with finite strategy sets and specifies which is the outcome that arises from each pure strategy profile.[3] It could identify, for instance, two ending nodes of an extensive-form game with the same outcome. Govindan and McLennan (2001) give an example of a game-form such that, in an open set of utilities over outcomes, produces infinitely many equilibrium distributions on outcomes. In view of such a negative result, we have to turn to specific classes of games to seek for positive results regarding the generic determinacy of the Nash equilibrium concept. For some examples, see Park (1997) for sender–receiver games, and De Sinopoli (2001) and De Sinopoli and Iannantuoni (2005) for voting games.

This paper studies the generic determinacy of the Nash equilibrium concept when individual payoffs depend on the network connecting them. The network literature has been fruitful to describe social and economic interaction. See for

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[1] By *outcomes* we mean the set of physical or economic outcomes of the game (i.e. the set of different economic alternatives that could be found after the game is played) and not the set of probability distribution induced by equilibria. We will refer to the latter concept as the set of equilibrium distributions.

[2] Harsanyi (1973) actually proves that the set of Nash equilibria is finite for a generic assignment of payoffs to pure strategy profiles.

[3] More generally, it specifies a probability distribution on the set of outcomes. Game-forms are formally defined in Section 2.2.

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instance Jackson and Wolinsky (1996), Kranton and Minehart (2001), Jackson and Watts (2002), or Calvo-Armengol (2004). It is, therefore, important to have theories that explain how such networks form. Different network-formation procedures have been proposed. For a comprehensive survey of those theories the reader is referred to Jackson (2005).

One of the most used network-formation games was suggested by Myerson (1991) and it can be described as follows. Each player simultaneously proposes a list of players with whom she wants to form a link and a direct link between two players is formed if and only if both players agree on that. Calvo-Armengol and İlkilic (2007) study this game to provide a connection between pairwise-stability, a network equilibrium concept defined in Jackson and Wolinsky (1996), and proper equilibrium, a noncooperative equilibrium refinement due to Myerson (1978).

This game is simple and intuitive, however, since it takes two players to form a link, a coordination problem arises which makes the game exhibit multiplicity of equilibria. This fact constitutes a common concern in network theory. Nevertheless, we can prove that even though a network-formation game may have a large number of equilibria, generically, the set of probability distributions on networks that they induce is finite.

The network-formation game is formally presented in the next section. Section 3 discusses an example. Section 4 contains the main result and its proof. To conclude, Section 5 discusses some remarks, extensions of the result to other network-formation games and a related result for the extensive-form game of network-formation introduced by Aumann and Myerson (1989).

2. Preliminaries

Given a finite set A, denote as $\mathcal{P}(A)$ the set of all subsets of A, and as $\Delta(A)$ the set of probability distributions on A.

2.1. Networks

Given a set of players $N = \{1, \dots, n\}$, a *network* g is a collection of direct links. A direct link in the network g between two different players i and j is denoted by $ij \in g$. For the time being we focus on undirected networks. In an undirected network $ij \in g$ is equivalent to $ji \in g$.[4,5]

The set of i's direct links in g is $L_i(g) = \{jk \in g\colon\ j = i \text{ or } k = i\}$. The complete network g^N is such that $L_i(g^N) = \{ij\colon\ j \neq i\}$, for all $i \in N$. In g^N player i is directly linked to every other player. The set of all undirected networks on N is $\mathcal{G} = \mathcal{P}(g^N)$.

Each player i can be directly linked with $n-1$ other players. The number of links in the complete network g^N is $n(n-1)/2$, dividing by 2 not to count links twice. Since $\mathcal{G}$ is the power set of g^N, it has $2^{n(n-1)/2}$ elements.

2.2. Game-forms

A *game-form* is given by a set of players $N = \{1, \dots, n\}$, nonempty finite sets of pure strategies $S_1, \dots, S_n$, a finite set of outcomes Ω, a function $\theta : S \to \Delta(\Omega)$, and utilities defined over the outcome space Ω, that is, $u_1, \dots, u_n : \Omega \to \mathbb{R}$. Once we fix N, $S_1, \dots, S_n$, Ω, and θ, a game-form is given by a point in $(\mathbb{R}^\Omega)^N$.

Utility functions $u_1, \dots, u_n$ over Ω induce utility functions $v_1, \dots, v_n$ over $S = S_1 \times \cdots \times S_n$ according to $u_1 \circ \theta, \dots, u_n \circ \theta$. Hence, every game-form has associated its finite normal form game.

2.3. The network-formation game

The set of players is N. All players in N simultaneously announce the set of direct links they wish to form. Formally, the set of player i's pure strategies is $S_i = \mathcal{P}(N \setminus \{i\})$. Therefore, a strategy $s_i \in S_i$ is a subset of $N \setminus \{i\}$ and is interpreted as the set of players other than i with whom player i wishes to form a link. Mutual consent is needed to create a direct link, i.e., if s is played, ij is created if and only if $j \in s_i$ and $i \in s_j$.

We can adapt the previous general description of game-forms to the present context in order to specify the game-form that structures the network-formation game. Let the set of players and the collection of pure strategy sets be as above. The set of outcomes is the set of undirected networks, i.e., $\Omega = \mathcal{G}$. The function θ is a deterministic outcome function, formally, $\theta : S \to \mathcal{G}$. Given a pure strategy profile, θ specifies which network is formed respecting the rule of mutual consent to create direct links. Utilities are functions $u_1, \dots, u_n : \mathcal{G} \to \mathbb{R}$. Once the set of players N is given, the pure strategy sets are automatically created and the network-formation game is defined by a point in $(\mathbb{R}^{\mathcal{G}})^N$.

If players other than i play according to $s_{-i} \in S_{-i}$,[6] the utility to player i from playing strategy s_i is equal to $v_i(s_{-i}, s_i) = u_i(\theta(s_{-i}, s_i))$.

[4] In a directed network, if i and j are two different agents, the link ij is different from the link ji. This two links can be regarded as different if, for instance, they explain which is the direction of information, or which is the player who is sponsoring the link.

[5] In graph theory, the kind of networks defined here are called *simple undirected graphs*.

[6] $S_{-i} = \prod_{j \neq i} S_j$.