Available online at www.sciencedirect.com
SCIENCE DIRECT

Nonlinear Analysis 64 (2006) 1415 – 1436

Nonlinear Analysis

www.elsevier.com/locate/na

Fixed point theorems and their applications to theory of Nash equilibria

S. Heikkilä[a,*], K. Reffett[b]

[a] Department of Mathematical Sciences, University of Oulu, Box 3000, Oulu 90014, Finland
[b] Department of Economics, WP Carey School of Business, Arizona State University, Tempe AZ 85287, USA

Received 30 June 2005; accepted 30 June 2005

Abstract

In this paper we prove fixed point theorems for set-valued mappings in products of posets. Applications to the theory of Nash equilibria are presented.

MSC: 47H04; 47H10; 91A10; 91A13; 91A44

Keywords: Fixed point; Set-valued mapping; Poset; Product space; Generalized iterations; Increasing; Ascending; Multi-player game; Nash equilibrium

1. Introduction

In this paper we apply fixed point results derived in [7] by a generalized iteration method to prove fixed point theorems for set-valued mappings in partially ordered sets (posets) and in their products. The main purpose is to develop tools for the study of Nash equilibria of the following multi-player game (cf. [15, Chapter 10]).

Let I be a set of players and T be a set of exogenous parameters reflecting the environment in which the players operate. Denote by X_i the *strategy set* of player i. Given a strategy $x = \{x_i\}_{i \in I} \in X = \prod_{i \in I} X_i$ of the players and $i \in I$, denote $x = (x_{-i}, x_i)$, where $x_{-i} = \{x_j\}_{j \in I \setminus \{i\}}$ is the strategy of other players. For all fixed $x \in X$ and $t \in T$, let a subset $X_i^{x,t}$

* Corresponding author. Tel.: +358 85531736; fax: +358 85531730.
E-mail address: sheikki@cc.oulu.fi (S. Heikkilä).

doi:10.1016/j.na.2005.06.043

of X_i denote the set of player i's *feasible replies* to (x, t), and let an element $u_i^t(x)$ of a poset $W_i = (W_i, \preccurlyeq_i)$ denote a *utility* to player $i \in I$. We say that a feasible reply y_i of player i to $(x, t) \in X \times T$ is *optimal* if y_i maximizes the utility $u_i^t(y)$ over $y_i \in X_i^{x,t}$ when $y_{-i} = x_{-i}$. A strategy $x = \{x_i\}_{i \in I}$ is called a *Nash equilibrium* for $t \in T$ if for each $i \in I$ the ith component x_i of x is an optimal reply of player i to (x, t).

Denoting by $F_i^t(x)$ the set of all optimal replies of player i to $(x, t) \in X \times T$, the above definitions imply that each Nash equilibrium for $t \in T$ is a fixed point of the set-valued mapping $F^t = \{F_i^t\}_{i \in I}$ whose components have values $F_i^t(x)$. Each fixed point of F^t is in turn a fixed point of a single-valued selection mapping $S^t = \{S_i^t\}_{i \in I} : X \to X$ with $S_i^t(x) \in F_i^t(x)$, $i \in I$, $x \in X$.

The above consideration yields the following plan of the paper. In Section 2 we provide basic abstract fixed point results for single-valued self-mappings of posets, proved in [7] by a generalized iteration method, and apply them to derive fixed point results for set-valued mappings F^t on posets, by assuming the existence of different kinds of selections for F^t. The results of Section 2 are then applied in Sections 3 and 4 to the case when X is a product of nonempty posets. Examples are given to show that the obtained results are not consequences of fixed point theorems in chain complete posets or in complete lattices (cf., e.g., [1–3,9,11–13,15,17]. In Section 5 we consider cases where fixed points are computable.

The fixed point results of Sections 3 and 4 will be so formulated that to every one of them there corresponds an existence result for Nash equilibria of the above defined multi-player game when the values $F_i^t(x)$ of each component of F^t are sets of all optimal replies of player i to $(x, t) \in X \times T$, provided that they are nonempty. In Section 6 we present conditions for the sets $X_i^{x,t}$ of feasible replies and for the utility mappings u_i^t which are sufficient for the existence of Nash equilibria. For instance, we prove that the greatest Nash equilibrium $\overline{x}^t$ exists and maximizes each utility $u_i^t(x)$ over all Nash equilibria for every $t \in T$ if each X_i is an order-bounded and order-closed subset of an ordered normed space E_i with regular order cone, if each $X_i^{x,t}$ is closed and directed upwards, if each mapping $x \mapsto X_i^{t,x}$ is increasing upwards, and if each utility mapping u_i^t is increasing.

Compared, e.g., to [10,14–16], the framework is more general in the sense that the strategy posets X_i's need not be complete lattices, not even lattices, and that the utility mappings u_i^t are poset-valued. These extensions provide new possibilities for the study of optimal equilibrium policies of various enterprises, for instance in economics. In our last example we apply an obtained theoretical result to a price game model presented in [15]. This example contains a numerical part, where an iteration method presented in Section 5 is applied.

2. Preliminaries

In this section we present basic fixed point results for single-valued mappings in posets, and apply them to derive fixed point results for set-valued mappings by reducing them to single-valued cases with the help of selection mappings.

A subset W of a poset Y is called *well-ordered* if each nonempty subset of W has the least element. W is said to be *inversely well-ordered* if each nonempty subset of W has the