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A note on the existence of Berge and Berge–Nash equilibria ☆

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Abstract

This paper deals with the problem of existence of Berge and Berge–Nash equilibria. Abalo and Kostreva have proved existence theorems of Berge and Berge–Nash equilibria for S-equi-well-posed and (S, σ)-equi-well-posed games, namely, Theorems 3.2–3.3 [Abalo, K.Y., Kostreva, M.M., 1996. Fixed Points, Nash Games and their Organization. Topological Methods in Nonlinear Analysis 8, 205–215.]. In this paper we show that the assumptions of these theorems are actually not sufficient for the existence of Berge equilibrium. We then propose a new version of these theorems.

Keywords: Equi-well-posed games; Berge equilibrium; Nash equilibrium; Fixed point

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1. Introduction and definitions

The basic idea of Berge equilibrium goes back to the notion of equilibrium for a partition K with respect (relative) to a coalition S introduced by Berge (1957). Indeed, using this notion, Zhukovskii introduced the Berge equilibrium (Zhukovskii, 1994) as an alternative solution for a non-cooperative game when it has no Nash equilibrium (Nash, 1951) or when it has several Nash

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equilibria. A Berge equilibrium is interpreted as follows (Zhukovskii, 1994). If a player chooses his strategy from a Berge equilibrium then, given this choice, he gets the maximum payoff if all the remaining players choose their strategies from this equilibrium. Berge equilibrium is quite different from Nash equilibrium. Indeed, a player receives his maximum payoff, given that the remaining players have chosen their strategies in a Nash equilibrium, if he chooses his strategy from the same Nash equilibrium. In Abalo and Kostreva (2005, 1996a,b), they give a more general definition of Berge equilibrium. They also provide a theorem of existence of this equilibrium, namely, Theorem 3.2 (Abalo and Kostreva, 1996a,b) for *S-equi-well-posed games*. The approach in this theorem is based on an earlier theorem of Radjef (Radjef, 1988) providing an existence theorem of Berge equilibrium in the sense of Zhukovskii (Zhukovskii, 1994). Furthermore, in Abalo and Kostreva (1996a,b) a theorem (Theorem 3.3) of existence of Berge equilibrium that is also Nash equilibrium or Berge–Nash equilibrium is established for (S, σ)-*equi-well-posed games*. In fact, the assumptions given in Theorems 3.2 and 3.3 are not sufficient for the existence of Berge equilibrium. Indeed, in this paper we first construct a simple game that satisfies the assumptions of Abalo and Kostreva's Theorem 3.2 (Abalo and Kostreva, 1996a,b) without Berge equilibrium in the sense of Zhukovskii, which is a special case of Berge equilibrium in the sense of Abalo and Kostreva. Then, we provide a new condition that overcomes the difficulty in this Theorem. Next, we construct another game verifying the assumptions of Abalo and Kostreva's Theorem 3.3 (Abalo and Kostreva, 1996a,b) without Berge equilibrium in the sense of Zhukovskii, hence without Berge–Nash equilibrium. The same problem can be pointed out for the Radjef's Theorem of existence of Berge equilibrium in the sense of Zhukovskii (Radjef, 1988). We do not provide any counterexample for it is a special case of Abalo and Kostreva's Theorem 3.2. In fact, the original subtle mistake comes from Radjef 's Theorem (Radjef, 1988), Abalo and Kostreva used this theorem without noticing it. In the papers (Abalo and Kostreva, 2005, 2004) some theorems of existence of Berge equilibrium for games having an S-system or an (S, R, M)-system are provided. In our paper (Nessah et al., 2007), we have shown that these theorems are not sufficient for the existence of Berge equilibrium and provided their corrected versions. In the present paper a similar approach is used to establish the results.

The paper is organized as follows. We first present the Berge equilibrium both in the sense of Abalo and Kostreva and in the sense of Zhukovskii. In Section 2, we provide a necessary and sufficient condition for the existence of Berge equilibrium in the sense of Zhukovskii. Section 3 presents the Abalo and Kostreva's Theorems 3.2 and 3.3. In Section 4, we first give a counterexample of a game satisfying assumptions of Theorem 3.2 and having no Berge equilibrium. Then, we explain why assumptions of Theorems 3.2 are not sufficient, propose a new condition to overcome the problem and a corrected version of this theorem. We end the paper by giving another example of a game verifying the assumptions of Abalo and Kostreva's Theorem 3.3 without Berge equilibrium, hence without Berge–Nash equilibrium. We also provide a corrected version of this theorem.

Consider the following non-cooperative game in normal form

$$G = (X_i, f_i)_{i \in I} \tag{1}$$

where $I=\{1,\ldots, n\}$ is the set of players; $X = \prod_{i\in I} X_i$ is the set of strategy profiles of the game where X_i is the set of strategies of player i; $X_i \subset E_i$, E_i is a vector space; f_i: $X \rightarrow \mathbb{R}$ is the payoff function of player i.

For each coalition $K \subset I$, we denote by $-K$ the set $\{i \in I$ such that $i \notin K\}$ of the coalition $I-K$, if K is reduced to a singleton $\{i\}$, we denote by $-i$ the set $-\{i\}$ and by $X_K \prod_{i\in K} X_i$ the set of