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Bayes–Nash equilibria of the generalized second-price auction [☆]

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ABSTRACT

We develop a Bayes–Nash analysis of the generalized second-price (GSP) auction, the multi-unit auction used by search engines to sell sponsored advertising positions. Our main result characterizes the efficient Bayes–Nash equilibrium of the GSP and provides a necessary and sufficient condition that guarantees existence of such an equilibrium. With only two positions, this condition requires that the click–through rate of the second position is sufficiently smaller than that of the first. When an efficient equilibrium exists, we provide a necessary and sufficient condition for the auction revenue to decrease as click–through rates increase. Interestingly, under optimal reserve prices, revenue increases with the click–through rates of all positions. Further, we prove that no inefficient equilibrium of the GSP can be symmetric. Our results are in sharp contrast with the previous literature that studied the GSP under complete information.

1. Introduction

Over the last few years a new multi-unit auction format known as the generalized second-price (GSP) auction has received much attention from economists. This auction format has been applied to diverse problems such as routing in fixed and wireless networks (see Su et al., 2010) and the allocation of capacity in electricity markets (see Gergen et al., 2008 and Schne, 2009). Most remarkably, search engines use the GSP to sell sponsored advertising links in the Internet, with revenues that exceeded 20 billion dollars industry-wide in 2009. According to the simplest version of this mechanism, each advertiser (bidder) submits one bid that represents his willingness to pay for a click in his sponsored link. Advertisers are then ranked in decreasing order of bids and sponsored links are assigned according to this ranking. Payments are determined according to a "next-bid" rule: Each advertiser pays for a click in his link the bid submitted by the advertiser immediately below him in the sponsored advertising list.

The GSP has been extensively studied in a complete information setting. Aggarwal et al. (2006), Varian (2007) and Edelman et al. (2007) are the first to derive complete information Nash equilibria of this auction. They show that, although its obvious similarities with the Vickrey auction, truthful bidding does not constitute an equilibrium of the GSP.

In an incomplete information setting, Edelman et al. (2007) modeled the GSP as an ascending auction for multiple goods (they call it the Generalized English auction). Their main result states that the unique perfect Bayesian equilibrium of this game implements the same payments as the Vickrey–Clark–Groves mechanism.

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Quite surprisingly, little is known about the Bayes–Nash equilibria of the GSP, where bids are submitted simultaneously and advertisers have private values per click which are position-independent. As Lahaie et al. stated in 2007, "To date nothing is known about the Bayesian equilibrium of the GSP auction".[1] In this paper, we develop a complete Bayes–Nash analysis of this mechanism.

Our main result characterizes the efficient Bayes–Nash equilibrium (simply "equilibrium" from now on) of the GSP and provides a necessary and sufficient condition that guarantees existence of such an equilibrium. The proof proceeds by using the integral-form envelope theorem to express the expected payment of a bidder with value per click v as a function of his probabilities of obtaining each position in an efficient equilibrium. Following the payment rule of the GSP, we derive an alternative expression for the expected payment of a bidder that depends on the symmetric bidding function employed by all bidders. In equilibrium, these expressions have to be equal for every valuation v. This condition leads to an integral equation (technically, a Volterra equation of the second kind) that any efficient equilibrium bidding function has to satisfy. We apply results from the theory of integral equations to show that there is a unique candidate bidding function that solves this integral equation, and derive its solution analytically. Using well-known results from Monotone Comparative statics, we show that an efficient equilibrium exists if and only if the candidate bidding function that solves the integral equation is strictly monotone at all possible valuations.

In the simple case where only two advertising positions are for sale, the candidate bidding function is strictly monotone (and an efficient equilibrium exists) if and only if the click–through rate of the second position is sufficiently smaller than that of the first. Intuitively, as the click–through rate of the second position approaches that of the first, obtaining the second position becomes a better deal for advertisers, since payments per click are lower (the third instead of the second highest bid) and click–through rates are similar. As a consequence, advertisers with high valuations have the incentive to shade their bids in order to be ranked second (rather than first) and obtain the second position. When click–through rates are close enough, bid shading is so intense that the bidding function is no longer monotonic, and the efficient equilibrium breaks down.

The bid shading phenomenon has interesting implications on how varying click–through rates affects the total revenue produced by an efficient equilibrium of the GSP. To obtain intuition, take the case of two positions and consider an increase in the click–through rate of the second position. There are two effects to consider: First, as payments are per click, revenue tends to increase as the number of clicks for sale increases (we call it the *supply effect*). Second, as click–through rates converge, advertisers with high values per click strategically shade their bids (we call it the *strategic effect*), and revenue tends to decrease. As it unfolds, we derive a necessary and sufficient condition on the distribution of values per click for the strategic effect to dominate the supply effect. Intuitively, this condition captures the notion that the distribution of advertisers' values per click is concentrated on high values, in which case the strategic effect is stronger than the supply effect; and total revenue goes down even though the number of clicks for sale is higher. This intuition readily generalizes to any number of positions.

Interestingly, with optimal reserve prices, we show that the search engine's revenue increases with the click–through rates of all positions (reversing the previous result on the GSP without reserve prices). Intuitively, reserve prices work to remove advertisers with low values per click from the auction. Hence, advertisers with high values per click can reduce their bid shading and still have a fair chance of getting lower positions. As a consequence, the supply effect unambiguously dominates the strategic effect; and the search engine's revenue can only grow with the number of clicks for sale in the auction.

Next, we turn to investigate the existence of inefficient equilibria of the GSP. We show that every equilibrium with symmetric strategies of the GSP is outcome equivalent to some monotone pure strategy equilibrium, and conclude that no inefficient equilibrium of the GSP can be symmetric. Our analysis reveals that the GSP is an interesting (yet simple) example of a game that satisfies the single-crossing condition applied by Athey (2001) and McAdams (2003) to discrete games, but still fails to admit (for a wide range of parameter values) a monotone Bayes–Nash equilibrium when action and type spaces are continuous.[2]

1.1. Related literature

Under complete information, Edelman et al. (2007) and Varian (2007) select among all equilibria that satisfy the envy-free criterion the one that produces the lowest revenues to the auctioneer. The envy-free criterion requires that no advertiser wishes to change positions and payments with some advertiser ranked above him. Cary et al. (2007) select the same equilibrium based on the convergence of myopic best responses. Edelman and Schwarz (2010) provide a different rationale for selecting this equilibrium based on the non-contradiction criterion (NCC). An equilibrium fails the NCC if it generates greater revenue (in expectation) than any equilibrium of an associated dynamic game.

Still under complete information, Borgers et al. (2008) allow advertisers to have position-specific values per click.[3] They extend the notion of envy-free equilibrium to this more general setting and show that weak dominance does not select the

[1] See the related literature for an account of recent work about the GSP under complete and incomplete information.

[2] See also Reny (2009).

[3] Milgrom (2010) considers an extension of the GSP where bidders are allowed to submit multiple bids, one for each position. Alternatively, Babaioff and Roughgarden (2009) measure the complexity of a payment rule by relating the profile of bids and the formation of prices for each position. Constantin et al. (2010) propose a variation of the GSP that allows each bidder to express how his value per click is affected by the identity of the other advertisers appearing in the sponsored list.