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Comparing approximation techniques to continuous-time stochastic dynamic programming problems: Applications to natural resource modelling

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ABSTRACT

Dynamic programming problems are common in economics, finance and natural resource management. However, exact solutions to these problems are exceptional. Instead, solutions typically rely on numerical approximation techniques which vary in use, complexity and computational requirements. Perturbation, projection and linear programming approaches are among the most useful of these numerical techniques. In this paper, we extend the parametric linear programming technique to include continuous-time problems with jump-diffusion processes, and compare it to projection and perturbation techniques for solving dynamic programming problems in terms of computational speed, accuracy, ease of use and scope. The comparisons are drawn from solutions to two fisheries management problems – a unidimensional model of optimal harvest and a multidimensional model for optimal marine reserve size. Available computer code illustrates how each technique solves these problems and how they can be applied to other comparable problems in natural resource modelling.

1. Introduction

Stochastic optimal control problems are used extensively in economics, finance and natural resource modelling. These problems normally involve optimising a life-time (discounted) value functional, subject to an allowable transition of state variables that can be controlled by a decision maker. Time in these problems can be modelled as a continuous or discrete variable, with the transition state governed by either differential or difference equations. The choice of a discrete or continuous-time framework depends on the setting and is usually determined on a case-by-case basis (Doraszelski and Judd, 2010). In many economic problems, modelling time as a discrete or continuous variable does not alter any of the important qualitative properties of the model, hence the choice of the time framework mainly depends on the convenience it provides to modellers, and occasionally on the data available to support the model.

However, in other situations, the choice between a discrete or continuous setting is crucial. For example, a discrete-time framework is more appropriate for seasonal, delayed or lagged effects, as illustrated in Clark (1976). On the other hand, modelling a diffusion process from a high density source to a low density sink in discrete-time may cause 'unexpected overshooting', i.e., after one period of time the diffusion process may make the sink more dense than the source. In this case, a continuous-time framework is more appropriate since the gradual adjustment over time will prevent such cases. In problems where discrete-time approximations do work well, and where 'overshooting' is not an issue, extensions to large state-spaces (e.g., Nicol and Chades, 2010) can generate a significant advantage over continuous-time methods.

Both discrete and continuous-time problems can be approached with the 'Principle of Optimality' introduced by Bellman (1957). In discrete-time, the Principle of Optimality leads to a so-called Bellman equation. In a continuous-time framework, the Principle of Optimality leads to Hamiltonian-Jacobi-Bellman (HJB) equation.

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Bellman and HJB equations can generate exact solutions in only a very few cases and normally solutions must instead rely on approximation techniques.

In general, the approximation techniques for HJB equations are more restrictive than for Bellman equations. The most common techniques to approximate a Bellman equation are value and policy function iterations. These are well-established and stable techniques which rely on the convergence of an iteration process. However, they do not work well for continuous-time problems. Theoretically, one can model continuous-time as a 'discretized-time' problem, with very small time intervals, and the use of value or policy function iterations. Unfortunately, as is well-known and detailed in Judd (1998) and Doraszelski and Judd (2010), the appropriate discount factor will then be close to unity causing the iteration process to converge very slowly.

Solving a HJB equation numerically usually relies on parametric approximation techniques which try to produce an analytical formula in the form of a linear combination of some pre-determined basis functions.[1] Three parametric techniques which have been applied to optimal control problems are the perturbation, projection and linear programming techniques. The perturbation and projection methods are well-known in economics due to the works of Judd and Guu (1993), Judd (1996, 1998), Gaspar and Judd (1997) and Arruda and DoVal (2006). Linear programming approaches to dynamic problems attract far less attention, and their application to HJB equations has only recently been introduced by the mathematicians, Han and Roy (2009), although the original idea can be dated back to Manne (1960) and Ross (1970).

While there are at least two papers that compare various numerical methods in discrete-time dynamic programming, namely Taylor and Uhlig (1990) and Aruoba et al. (2006), comparisons of techniques for continuous-time problems are scant and usually brief. Gaspar and Judd (1997), for example, briefly indicate that the projection technique is typically much slower than a perturbation method, while Judd (1998) simply points out that the projection technique is a general approach to functional equations, hence naturally applicable to continuous-time dynamic programming problems. Han and Roy (2009), while introducing the basic structure of the linear programming technique, do not explore computational issues, and only mention that this new technique is efficient and convenient.

In this paper, we compare the three parametric techniques for HJB equations to highlight their relative strengths and weaknesses to continuous-time problems with jump-diffusion processes – processes especially applicable to problems in natural resource modelling. Our comparison differs from the two previous works that compare these techniques for discrete-time problems in at least three ways. First, the comparison of the techniques' performance is in an applied framework, illustrated through two useful numerical case studies in fisheries management. The first model is unidimensional, with one control and one state variable. The second model is multidimensional, with one control and two state variables, and includes a diffusion from a source to a sink, thus underscoring the need for a continuous-time framework. Each problem is solved with the three techniques using the same workstation and coding platform, tested in both a PC and MAC environment, with MATLAB R2009 (see the Appendix A on platforms, programs and available programming code). The evaluation of their performance is partly based on the size of approximation errors and computation time, and underscores how the relative superiority of the various techniques changes with respect to the dimension of the problem being solved.

Second, we are not solving a basic classical model to compare techniques, where all qualitative properties are already known. Instead, the numerical case studies in this paper have highly generalised non-linear structures where return functions are dependent on both state and control variables and uncertainty components are state-dependent. These models are increasingly used in applications to biosecurity and fisheries economics, along with other natural resource problems, where not only qualitative properties but more exact numerical results are important to researchers.

Third, as a natural question arising from any exercise that 'picks a winner', we also compare techniques taking into account not only approximation quality and computation time, but other considerations that influence the choice of a technique in practice. For example, all three techniques have different approaches to HJB equations so their software package requirements vary. Another consideration is the fact that each technique may have variants that are more or less efficient to a particular problem. Although it is impossible to report all variants, we nevertheless provide a general guide to the relative advantages and disadvantages of each of these three techniques, their overall ease of use and their scope.

The remainder of the paper is organised as follows. In Section 2, we formulate a generalised dynamic optimisation problem and specify the corresponding HJB equation. The uncertainty components in our generalised formulation are not only driven by Brownian motion but also a Poisson jump-diffusion process, a standard instrument to model randomly discontinuous jumps.[2] Poisson diffusions have been used to model events and generate key results in many studies, for example, in the models of technological progress in Aghion and Howitt (1992) and Walde (1999), interest rate movements in Das (2002) and Piazzesi (2005), and negative shocks to fish stocks in Grafton et al. (2005).

In Section 3, we briefly describe the three techniques to solve the HJB equation for the readers' convenience. The description of the projection and perturbation techniques is especially brief, since they are relatively well-known. Further details for these techniques can be found in Judd (1998). The parametric linear programming technique, on the other hand, is described much more carefully, since it is relatively new. We also provide a theorem, extended from Han and Roy (2009), to accommodate models with Poisson jump-diffusions, thus providing a theoretical basis for this new technique.

Sections 4 and 5 are devoted to the numerical case studies. Each problem is introduced and solved, with reported approximation errors and computation time. With each technique, we solve the problems in the most plain manner, putting aside complicated variants that can be applied to a particular situation. Section 6 addresses the question of 'which technique wins in practice' and Section 7 concludes.

2. A generalised stochastic optimal control problem in a continuous-time setting

We begin with a general optimal control problem in a continuous-time setting. The problem is to identify the maximum value function $V(s)$ and/or the optimal profile of the control variables $c(t)$ such that:

[1] Approaching HJB equations as a partial differential equation and solving via finite difference schemes is often complicated and inconvenient due to the absence of clear boundary conditions (see Hedlund (2003) for a basic illustration of this).

[2] Though involving discontinuous jumps, Poisson processes are considered as continuous diffusions with respect to time as the probability of the discontinuous jump occurring in a time interval converges to zero when the time interval approaches zero.